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Some optimal error estimates of biharmonic problem using conforming finite element ☆

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Abstract

In this paper, the $W^{2,p}$-error and L^p-error estimates of conforming finite element methods for biharmonic problem are established via weight function technique, where $2 \leqslant p \leqslant +\infty$. Finally, the numerical experiments are provided.

Keywords: Optimal error estimates; Biharmonic problem; Finite element approximation

1. Introduction

About 30 yeas ago, many scholars began to investigate the max–norm error estimates of second order Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \subset \mathrm{R}^2$$

with Dirichlet boundary

$$u = 0 \quad \text{on } \partial\Omega$$

when Ω was partitioned by regular mesh and irregular mesh, such as [1–11] and references therein. First, Nitsche in [1] gave

$$\|u - u_h\|_{0,\infty} \leqslant ch^2 |\ln h|^{\frac{3}{2}} |u|_{2,\infty} \tag{1}$$

and

$$\|u - u_h\|_{1,\infty} \leqslant ch |\ln h| |u|_{2,\infty} \tag{2}$$

by the method of weight function for linear finite element. But it is obvious that (1) and (2) are not of optimal order. Subsequently, Nitsche in [2] improved the estimate (1) and showed

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$$\|u-u_h\|_{0,\infty} \leqslant ch^2|\ln h||u|_{2,\infty}. \tag{3}$$

Fried in [3] gave an example based on radial symmetry which implied that the estimate (3) maybe was of optimal order. Hence, it is difficulty to remove the factor $|\ln h|$ in (3). However, the logarithmic factor in (2) can be removed. Rannacher and Scott in [4] improved the estimate (2) and obtained some optimal error estimates

$$\begin{cases} \|u-u_h\|_{1,p} \leqslant ch\|u\|_{2,p} & 2 \leqslant p \leqslant +\infty, \\ \|u-u_h\|_{0,q} \leqslant ch^2\|u\|_{2,q} & 2 \leqslant q < +\infty. \end{cases} \tag{4}$$

In particular, for high order finite element, the factor $|\ln h|$ in (1) and (3) disappeared and optimal max–norm error estimates were established

$$\|u-u_h\|_{0,\infty} \leqslant ch^2\|u\|_{2,\infty} \tag{5}$$

and

$$\|u-u_h\|_{1,\infty} \leqslant ch\|u\|_{2,\infty}. \tag{6}$$

See [1] for more details. Moreover, if the domain Ω was partitioned by irregular mesh, Scott in [5] showed that (5) was also true.

Similar results have been obtained for second order nonlinear problems. Frehse and Rannacher in [6] obtained

$$\|u-u_h\|_{0,\infty} \leqslant ch^2|\ln h||u|_{2,\infty}$$

for linear finite element approximation and in [7]

$$\|u-u_h\|_{0,\infty} \leqslant ch^m$$

for high order $m \geqslant 3$ finite element approximation. If $\Omega \subset \mathrm{R}^N, N \geqslant 3$, they in [6] proved

$$\|u-u_h\|_{0,\infty} \leqslant ch^2|\ln h|^{\frac{N}{4}+1}|u|_{2,\infty}.$$

We refer the reader to [8–12] for other similar results.

However, to our best knowledge, it is vacant for max–norm error estimate of high order problem. Therefore, in this paper, we deal with the biharmonic problem

$$\Delta^2 u = f(x) \quad \text{in } \Omega \subset \mathrm{R}^2$$

with Dirichlet boundary

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We will extend the method in [4] to fourth order problem and show some optimal estimates

$$\begin{cases} \|u-u_h\|_{2,p} \leqslant ch^2\|u\|_{4,p}, & 2 \leqslant p \leqslant +\infty, \\ \|u-u_h\|_{0,q} \leqslant ch^4\|u\|_{4,q}, & 2 \leqslant q < +\infty \end{cases} \tag{7}$$

by conforming finite element approximation. If $q = +\infty$, we will show

$$\|u-u_h\|_{0,\infty} \leqslant ch^3|u|_{4,\infty}, \tag{8}$$

which is not optimal. But the proof in regular triangular partition is much complicated than in quadrilateral partition. In particular, if $p = q = 2$, it is well-known that the estimate (7) is true (cf. [13]).

This paper is organized as follows: in Section 2, we give some preliminary knowledge and define finite element subspace when Ω is partitioned by regular quadrilaterals; in Section 3, a stability theorem is established; in Section 4, we will prove the error estimates (7) and (8); in Section 5, the case when Ω is decomposed into regular triangles is discussed; in Section 6, the numerical experiments are provided.

Throughout this paper, the symbol c always denotes a positive constant and is independent of h. Moreover, it maybe is different even in the same formulation.