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Weak co-coercivity and its applications in several algorithms for solving variational inequalities ☆

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Abstract

In this paper we investigate several algorithms for solving variational inequalities (VIs) and the zero inclusion problem, respectively. It is shown that invested algorithms are applicable in broader area. In addition, for the purpose of practical implementation, we further modify one of the algorithms discussed by adopting an Armijo-like search.

Keywords: Variational inequalities; Weak co-coercivity; Sub-strong monotonicity; Maximal monotone; Splitting method; Armijo-like search

1. Introduction

Consider the following variational inequality (VI) problem, which is to find an $x^* \in C$ such that:

$$\langle F(x^*), x - x^* \rangle + f(x) - f(x^*) \geqslant 0, \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty, closed convex subset of R^n, F is a continuous mapping from C into R^n, and f is a continuous convex function from C into R. In particular, if $f(x)$ is constant over C in (1.1), then we get the variational inequality problem in the common sense: to find $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geqslant 0, \quad \forall x \in C. \tag{1.2}$$

Due to their wide applications in many fields, variational inequalities have received much attention, and various iterative schemes have been proposed for solving them. The interested reader may consult the monograph by Facchinei and Pang [3] for details on this subject.

In several methods for VI, the co-coercivity of $F(x)$ is imposed to ensure the convergence of algorithms. In addition, the co-coercivity of the operator is also required in some algorithms solving the zero inclusion problem (see for example [3,4,6,9]). To our best knowledge, the co-coercivity is the weakest assumption for those algorithms without line search.

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Recently, Yang proposed a notion, called weak co-coercivity, and showed that it is weaker than co-coercivity and might take the role of the co-coercivity in the convergence analysis of some iterative schemes for VI [7,8].

The main purpose of the present paper is to further investigate several iterative schemes for solving VIs under the weak co-coercivity assumption. It turns out to be that the convergence of the algorithms discussed keeps valid under this milder assumption.

In addition, for the purpose of practical implement, we propose an improvement of the general scheme for VIs and show that it is a generalization of the extragradient projection method with stepsize search.

This paper is organized as follows. In Section 2, several properties of weakly co-coercive mappings are given. In Section 3, we prove the convergence of some general schemes for solving VIs under the weak co-coercivity assumption. In Section 4, we establish the convergence of the forward–backward splitting method for finding the zero of the operator $T(x)$ on R^n with the weak co-coercivity assumption. Finally, in Section 5 we make some further discussions and modify the algorithms for solving VIs by adopting an Armijo-like search.

2. Definitions and preliminary results

In this paper, we denote by $\|\cdot\|_2$ the 2-norm in R^n, and for any vector norm $\|\cdot\|$, the corresponding matrix norm is

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

Definition 1. A mapping F is called co-coercive on C, if there exists a positive constant α such that

$$\langle F(x) - F(y), x - y\rangle \geqslant \alpha\|F(x) - F(y)\|_2^2, \quad \forall x, y \in C. \tag{2.1}$$

Definition 2. A mapping F is called weakly co-coercive, if there is a positive continuous function $\alpha(x,y)$ on $C \times C$ such that

$$\langle F(x) - F(y), x - y\rangle \geqslant \alpha(x,y)\|F(x) - F(y)\|_2^2, \quad \forall x, y \in C. \tag{2.2}$$

It is easy to show that weak co-coercivity is a weaker condition than co-coercivity. In fact, if $\alpha(x,y)$ in (2.2) is a constant or has an infimum $\bar{\alpha} > 0$, then $F(x)$ is co-coercive. However, if C is unbounded and $\alpha(x,y)$ tends to zero as $\|x\|_2$ or $\|y\|_2$ approaches infinity, then $F(x)$ is not necessarily co-coercive. An illustrative example $F(x) = 1 - e^{-x}$ on R was given in [7], which was proved to be neither co-coercive nor strongly monotone.

Similarly, we can introduce the following definition.

Definition 3. The mapping F is sub-strongly monotone on C, if there exists a positive continuous function $\beta(x,y)$ on $C \times C$ such that

$$\langle F(x) - F(y), x - y\rangle \geqslant \beta(x,y)\|x - y\|_2^2, \quad \forall x, y \in C. \tag{2.3}$$

It is obvious that sub-strong monotonicity is weaker than strong monotonicity. Let us consider the function $F(x) = 1 - e^{-x}$ on R. Since

$$\langle F(x) - F(y), x - y\rangle = (e^{-y} - e^{-x})(x - y) \geqslant \min\{e^{-x}, e^{-y}\}(x - y)^2$$

we can see that $F(x) = 1 - e^{-x}$ is sub-strongly monotone on R with taking $\beta(x,y) = \min\{e^{-x}, e^{-y}\}$.

Proposition 2.1. *If a mapping F is sub-strongly monotone and Lipschitz continuous on C, then F is weakly co-coercive.*

This proposition can be proved straightforward from the definitions, but not vice versa.

Definition 4. A set-valued operator T on a Hilbert space $\mathscr{H}$ is called a maximal monotone operator [3], if T is monotone, i.e., $\forall x, y \in \mathscr{H}$, $\forall v \in T(x), \forall w \in T(y)$, $\langle v - w, x - y\rangle \geqslant 0$, and the graph $Gr\ T = \{(x, v) \in$