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The existence of countably many positive solutions for one-dimensional p-Laplacian with infinitely many singularities on the half-line ☆

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Abstract

We study the existence of countably many positive solutions of boundary value problems on the half-line for differential equations of second-order. The fixed-point index theory and a new fixed-point theorem in cones are used.

Keywords: Singularity; Multiple positive solutions; Multiple points boundary value problems; Fixed-point theorem; Half-line; Cone

1. Introduction

In this paper, we study the existence of countable many positive solutions for m-point boundary value problem with a p-Laplacian operator on a half-line

$$(\phi_p(u'))' + a(t)f(u(t)) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0, +\infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$ and α_i satisfy $\alpha_i \in [0, +\infty)$, $0 < \sum_{i=1}^{m-2}\alpha_i < 1$, $f \in C([0, +\infty), [0, +\infty))$, $a(t) : [0, +\infty) \to [0, +\infty)$ and has countably many singularities in $[1, +\infty)$.

The existence and multiplicity of positive solutions for linear and nonlinear ordinary differential equations and difference equations have been studied extensively. To identify a few, we refer the reader to see [2,6–10,12,14]. Recently, in paper [7] study the existence of positive solutions of the following singular

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three-point boundary value problems for one-dimensional under the condition that $f \in C([0,+\infty),[0,+\infty))$, $a(t):[0,1] \to [0,+\infty)$ and have countable many singularities on $(0,\frac{1}{2})$ p-Laplacian operator

$$\begin{cases} (\phi_p(u'))' + a(t)f(u(t)) = 0, & 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \beta u(\eta). \end{cases}$$

In [6,8] they also obtain countable many positive solutions by assuming $a(t)$ has infinitely many singularities in $(0,\frac{1}{2})$.

Seeing such a fact, we will think "whether or not we can obtain countable many positive solutions for the boundary value problems of differential equation on infinite intervals?" As a result, the goal of present paper is to fill the gap in this area. Obviously, using the similar method as [6–8] we can get the same conclusion when $a(t)$ has infinitely many singularities in $(\frac{1}{2},1)$ if we change corresponding conditions. So we only assume $a(t)$ has infinitely many singularities in $(1,+\infty)$ in this paper.

The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems on the half-line occur naturally in the study of radially symmetric solutions of nonlinear elliptic equations (see [4,13]) and various physical phenomena [3,11], such as unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, in determining the electrical potential in an isolated neutral atom. In all these applications, it is frequent that only solutions that are positive are useful. Recently there have been many papers investigated the positive solutions of boundary value problem on the half-line, see [1,15–18], they discuss the existence and multiplicity (at least three) positive solutions to nonlinear differential equation. However, to the best knowledge of the authors, there is no paper concerned with the existence of countable many positive solutions to m-point boundary value problems of differential equation on infinite intervals so far.

So in this paper, we use fixed-point index theory and a new fixed-point theorem in cones investigate the existence of countable solutions to boundary value problems (1.1) and (1.2).

We will assume that the following conditions are satisfied throughout this paper:

(C_1) $f \in C([0,+\infty),[0,+\infty)), f(0) \not\equiv 0$ on any subinterval of $(0,+\infty)$ and when u is bounded $f((1+t)u)$ is bounded on $[0,+\infty)$;

(C_2) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $1 < t_i < t_{i+1}$, $\lim_{i\to\infty} t_i = t_0 < +\infty, \lim_{t\to t_i} a(t) = \infty$, $i = 1,2,\ldots$, and

$$0 < \int_0^{+\infty} a(t)\mathrm{d}t < +\infty, \quad \int_0^{+\infty} \phi_p^{-1}\left(\int_\tau^{+\infty} a(s)\mathrm{d}s\right)\mathrm{d}\tau < +\infty.$$

Moreover $a(t)$ does not vanish identically on any subinterval of $[0,+\infty)$.

The plan of the paper is as follows. In Section 2, for the convenience of the reader we give some definitions. In Section 3, we present some lemmas in order to prove our main results. Section 4 is developed to presenting and proving our main results.

2. Some definitions and fixed-point theorems

In this section, we provide some background definitions cited from cone theory in Banach spaces.

Definition 2.1. Let $(E,\|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied

(a) if $y \in P$ and $\lambda \geqslant 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leqslant$, that is, $x \leqslant y$ if and only if $y - x \in P$.