Applied Mathematics and Computation 204 (2008) 395–407

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Lyapunov type operators for numerical solutions of PDEs

Anouar Ben Mabrouk [a,b,*], Mekki Ayadi [b]

[a] Computational Mathematics Laboratory, Department of Mathematics, Faculty of Sciences, 5019 Monastir, Tunisia
[b] Department of Computer Sciences, Higher Institute of Applied Sciences and Technology, Ibn Khaldoun, 4003 Sousse, Tunisia

ARTICLE INFO

Keywords:
Lyapunov equation
Lyapunov operator
NLS equation
Heat equation
Finite difference scheme
Von Neumann method
Stability analysis
Consistency
Convergence
Error estimates
Fixed point theory

ABSTRACT

In the present paper, numerical methods are developed to approximate the solutions of some evolutionary nonlinear problems. The continuous problems are transformed into some Lyapunov type equations and then analysed for existence, uniqueness, convergence, stability and error estimates. The main idea consists in applying Fourier analysis and Von Neumann criterion acting translation and scaling parameter methods to obtain contractive operators leading to fixed point theory.

1. Introduction

The paper is devoted to the development of numerical solutions of some nonlinear problems of the form:

$$\mathscr{L}(u) + f(u) = 0 \quad \text{in } \Omega \times (t_0, +\infty) \tag{1}$$

with boundary and initial conditions:

$$u(x, y, t_0) = u_0(x, y), \quad (x, y) \in \overline{\Omega}, \tag{2}$$

$$\frac{\partial u}{\partial n}(x, y, t) = 0, \quad ((x, y), t) \in \partial\Omega \times (t_0, +\infty). \tag{3}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, t_0 is a real parameter fixed as the initial time, f is a nonlinear function of u characterized by mixed nonlinearities. $\frac{\partial}{\partial n}$ is the outward normal derivative operator along the boundary $\partial\Omega$. u and u_0 are complex valued functions. u_0 is usually supposed to be sufficiently regular. In this paper, we consider two types of operators $\mathscr{L}$:

- The Schrödinger operator $\mathscr{S}(u(x, t)) = i\frac{\partial u}{\partial t} + \Delta u$ leading to Schrödinger equation.
- The Heat operator $\mathscr{H}(u(x, t)) = \frac{\partial u}{\partial t} - \Delta u$ leading to Heat equation.

Δ is the Laplace operator in $\mathbb{R}^d$. In the present work, we consider a continuous function f satisfying the assumption:

$$|f(u) \pm |u|^{p-1}u| \leqslant g_q(u), \tag{4}$$

where $p > 1$ is a real parameter and g_q is non-lipchitzian and locally q-Hölder continuous function, i.e.,

* Corresponding author. Address: Computational Mathematics Laboratory, Department of Mathematics, Faculty of Sciences, 5019 Monastir, Tunisia.
E-mail addresses: anouar.benmabrouk@issatso.rnu.tn (A.B. Mabrouk), mekki.ayadi@enis.rnu.tn (M. Ayadi).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2008.06.061

$$|g_q(x) - g_q(y)| \leqslant C_{\text{loc}}|x - y|^q, \quad C_{\text{loc}} \text{ and } 0 < q < 1. \tag{5}$$

Nonlinear Schrödinger and Heat equations are related to several phenomena such as, plasmas, optics, condensed matter physics, Heat conduction and Heat transfer. They then attracted the interest of researchers in physics and applied mathematics. Many studies have been brought for both theoretical and numerical treatments of such equations. However, there remains a lot of questions to be investigated. For backgrounds on such a subject, the readers are referred to [1,6–9,15]. We just recall that almost all these references have been restricted to the one-dimensional case especially numerical ones. In [5], higher dimensional models have been studied for nonlinear radial Schrödinger and Heat equations. In [5,6], some numerical schemes for Schrödinger and Heat equations are provided by transforming them to completely linear discrete algebraic systems.

The present work lies in the whole scope of numerical studies of PDEs essentially NLS and Heat equations. Numerical methods are developed to approximate the solutions of problems (1) with the operators $\mathscr{S}$ and $\mathscr{H}$ by replacing the time and the space partial derivatives by finite-difference replacements and the nonlinear term by a semi-linearized one in order to transform the initial boundary-value problems into quasi-linear algebraic systems. Then, the resulting methods are analysed for local truncation error and stability and we prove that the scheme is uniquely solvable and convergent. The main idea is to apply the Von Neumann method which consists in testing the impact of the proposed scheme on an isolated Fourier mode. Furthermore, some numerical examples are developed in order to validate our theoretical results. A more performant procedure than that of using tridiagonal systems is proposed to transform the discrete quasi linear problems into algebraic equations of the form $AX + XA^{\mathrm{T}} = B$ known as the Lyapunov equation leading to the study of the linear operator $\mathscr{L}(X) = AX + XA^{\mathrm{T}}$ defined on an appropriate matrix space. Here, A is a fixed matrix depending on the discretization method and on the problem parameters. We then focused on studying the operator $\mathscr{L}$ and searching for possible eigenvalues. The problem is transformed from a starting idea on resolving PDEs to a theoretical algebraic study of linear operators of the form $\mathscr{L}(X) = AX + XB$ where A and B are given matrices. We search for necessary and sufficient conditions on A and B to guaranty an invertible operator, and for eventual relationship between the eigenvalues of $\mathscr{L}$ and those of A and B. Unfortunately, when developing our ideas, some recent work dealing with the same algebraic problem appeared. See [14]. Almost all the questions that we have confronted with have been answered. This is unfortunate in some way but fortunate because it proves the impact of the question and let us to win time in improving and continuing our staring work about NLS and Heat equations. Nevertheless, we recall that almost all the studies that have been done about these equations except [1–6] have focussed on:

$$f(u) = \pm|u|^{p-1}u,$$

where $p > 1$. For more details on such equations, related applications, theoretical and numerical treatments, we refer to [7,10,12,13,16,17] and those therein. Here, the main difference is to consider nonlinear term $f(u)$ characterized by mixed concave and convex nonlinearities.

The paper is organized as follows. In the next section, some basic ideas from [14] on the invertibility of operators of the form $\mathscr{L}(X) = AX + XB$ are recalled. In Section 3, the discretization of (1)–(3) with the Schrödinger operator $\mathscr{S}$ is developed. The uniqueness of the solution, convergence of the scheme, the consistency and the stability of the method are analysed by applying the Von Neumann method. In Section 4, a quite similar numerical scheme is provided to transform the system (1)–(3) with the operator $\mathscr{H}$ into a quasi-linear discrete algebraic system. The uniqueness of the solution, convergence of the scheme, the consistency and the stability are discussed as usual. In Section 5, some numerical examples are exposed in order to validate the schemes. We focus on the mixed model:

$$f(u) = |u|^{p-1}u + \varepsilon|u|^{q-1}u.$$

The main idea is by transforming the discrete problems into Lyapunov type equations based on contractive operators $\mathscr{L}$. In fact, the first operators obtained are not contractive. So, we acted a translation parameter for the discrete problem associated to Schrödinger equation and a scaling one for Heat equation. Such actions yielded contractive operators allowing the application of fixed point theory.

2. On the Lyapunov operator

The resolution of the so-called Lyapunov equation:

$$AX + XA^{\mathrm{T}} = C, \tag{6}$$

where A and C are given matrices and X is an unknown one, is often of interest and has been the object of some studies until the 1960s. See [11]. However, even-though, the studies have been characterized by restrictive aspects. For example, in [11], a Kronecker product based method has been developed to resolve Eq. (6). The method consisted in a recursive scheme using the well known Caylay result and it gives explicit solutions in two- and three-dimensional cases. Such cases are not complicated and a verification or a computation can be proved with simpler and direct methods than the one in [11]. The equation has been re-considered by many authors due to its relation and application in PDEs, theory of stability and dynamical sys-