Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 198 (2008) 700–714
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Matrix transformations and compact operators on some new *m*th-order difference sequences ☆

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Abstract

We define some new sets of sequences the *m*th-order differences of which are α-bounded, convergent and convergent to zero, and apply the general methods in [E. Malkowsky, V. Rakočević, On matrix domains of triangles, Appl. Math. Comput. 189 (2) (2007) 1146–1163] to give Schauder bases for the latter two, determine their β-duals and characterize matrix transformations on them. Our results generalize those in [B. de Malafosse, The Banach algebra S_α and applications, Acta Sci. Math. (Szeged) 70 (1–2) (2004) 125–145] and improve those in [E. Malkowsky, S.D. Parashar, Matrix transformations in spaces of bounded and convergent difference sequences of order *m*, Analysis 17 (1997) 87–97]. We also establish identities and estimates for the Hausdorff measure of non-compactness of matrix operators from our spaces into the spaces of bounded, convergent and null sequences, and characterize the respective classes of compact operators. Some of these results generalize those in [E. Malkowsky, V. Rakočević, The measure of non-compactness of linear operators between spaces of *m*th-order difference sequences, Stud. Sci. Math. Hungar. 33 (1999) 381–391].

Keywords: Sequence spaces; Difference sequence; BK spaces; Matrix domains; Matrix transformations; Compact operators; Hausdorff measure of non-compactness

1. Some notations and preliminary results

We start with an introduction to the basic definitions and notations.

Let $(X, \|\cdot\|)$ be a normed space. Then the unit sphere and closed unit ball in X are denoted by $S_X = \{x \in X : \|x\| = 1\}$ and $\overline{B}_X = \{x \in X : \|x\| \leqslant 1\}$. If X and Y are Banach spaces then $\mathscr{B}(X, Y)$ is the set of all bounded linear operators $L : X \to Y$; $\mathscr{B}(X, Y)$ is a Banach space with the operator norm given by

☆ Research of both authors supported by the research project ♯144003 of the Serbian Ministry of Science, Technology and Development.
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0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.09.008

$\|L\| = \sup_{x\in S_X}\|L(x)\|$. In particular, if $Y = \mathbb{C}$ then we write X^* for the set of all continuous linear functionals on X with the norm given by $\|f\| = \sup_{x\in S_X}|f(x)|$.

Let ω denote the set of all complex sequences $x = (x_k)_{k=0}^\infty$. We write ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences, respectively, and cs and bs for the sets of all convergent and bounded series. Let e and $e^{(n)}$ $(n = 0, 1, \ldots)$ be the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$.

If x and y are sequences and X and Y are subsets of ω then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$, and $X^\beta = \bigcap_{x\in X} x^{-1} * cs$ and $X^\gamma = \bigcap_{x\in X} x^{-1} * bs$ for the β- and γ-duals of X; so $a \in X^\beta$ if and only if $\sum_{k=0}^\infty a_k x_k$ converges for all $x \in X$, and $a \in X^\gamma$ if and only if $(\sum_{k=0}^n a_k x_k)_{n=0}^\infty$ is bounded for all $x \in X$.

Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=0}^\infty$ and $A^k = (a_{nk})_{n=0}^\infty$ for the sequences in the nth row and the kth column of A, $A_n x = \sum_{k=0}^\infty a_{nk} x_k$, $Ax = (A_n x)_{n=0}^\infty$ (provided all the series $A_n x$ converge), and $X_A = \{x \in \omega : Ax \in X\}$ for the *matrix domain of A in X*. Also (X, Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$.

A sequence $(b_n)_{n=0}^\infty$ in a linear metric space X is called a *Schauder basis* if for every $x \in X$ there is a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b_n$.

The theory of BK spaces is of great importance in the characterization of matrix transformations between sequence spaces. A Banach space $X \subset \omega$ is a *BK space* if each projection $x \mapsto x_n$ on the nth coordinate is continuous. A BK space $X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=0}^m x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=0}^\infty \in X$.

Throughout, let $T = (t_{nk})_{n,k=0}^\infty$ be a triangle, that is $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1, \ldots)$, and S be its inverse. The inverse of a triangle exists, is unique and a triangle [18, 1.4.8, p. 9, 2, Remark 22 (a), p. 22].

We also need the following notations [5,12] for the definition of our sequence spaces.

Let m be a positive integer throughout. We define the operators $\Delta^{(m)}, \Sigma^{(m)} : \omega \to \omega$ by

$$(\Delta^{(1)}x)_k = \Delta^{(1)}x_k = x_k - x_{k-1}, \quad (\Sigma^{(1)}x)_k = \Sigma^{(1)}x_k = \sum_{j=0}^k x_j \ (k = 0, 1, \ldots)$$

and

$$\Delta^{(m)} = \Delta^{(1)} \circ \Delta^{(m-1)}, \Sigma^{(m)} = \Sigma^{(1)} \circ \Sigma^{(m-1)} \quad (m \geqslant 2).$$

The following results hold for $m \geqslant 1$ and $k = 0, 1, \ldots$ (see [13, p. 183]):

$$(\Delta^{(m)}x)_k = \sum_{j=0}^m (-1)^j \binom{m}{j} x_{k-j}, \tag{1.1}$$

$$(\Sigma^{(m)}x)_k = \sum_{j=0}^k \binom{m+k-j-1}{k-j} x_j \tag{1.2}$$

and

$$\Delta^{(m)} \circ \Sigma^{(m)} = \Sigma^{(m)} \circ \Delta^{(m)} = id, \quad \text{where } id \text{ is identity on } \omega. \tag{1.3}$$

We write Δ and Σ for the matrices with $\Delta_{nk} = (\Delta^{(1)}(e^{(k)}))_n$ and $\Sigma_{nk} = (\Sigma(e^{(k)}))_n$ for all n and k. So the operators $\Delta^{(1)}$ and $\Sigma^{(1)}$ are given by the matrices Δ and Σ. Similarly, the operators $\Delta^{(m)}$ and $\Sigma^{(m)}$ are given by the mth powers Δ^m and Σ^m of the matrices Δ and Σ. We note that, since Δ and Σ are triangles and inverse to one another, so are Δ^m and Σ^m.

If $X \subset \omega$ then we write $X(\Delta^m) = X_{\Delta^m}$. Our results generalize those obtained for the spaces $X(\Delta)$ in [1,7]; spaces of mth-order difference sequences were studied in [11,12,4,3,9].

Furthermore let $\mathcal{U} = \{u \in \omega : u_n \neq 0 \text{ for all } n\}$ and $\mathcal{U}^+ = \{u \in \omega | u_n > 0 \text{ for all } n\}$. Given $\alpha \in \mathcal{U}$, we write $1/\alpha = (1/\alpha_n)_{n=0}^\infty$. The spaces $s_\alpha^0 = (1/\alpha)^{-1} * c_0$, $s_\alpha^{(c)} = (1/\alpha)^{-1} * c$, $s_\alpha = (1/\alpha)^{-1} * \ell_\infty$ where $\alpha = (\alpha_n)_{n=0}^\infty \in \mathcal{U}^+$ were introduced and studied in [8,10]. We define the following sets for $\alpha \in \mathcal{U}$: