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Symmetry group and non-propagating solitons in the (2 + 1)-dimensional sine-Gordon equation

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Abstract

Starting from the compact version of the (2 + 1)-dimensional sine-Gordon equation, the symmetry group and then the Lie symmetries and the related algebra can be obtained via a simple combination of a gauge transformation of the spectral function and the transformations of the space time variables. Thereout, applying the group transformation theorem on a two-straight-line soliton solution, one can derive two types of meaningful non-propagating dromion lattice excitations.

Keywords: (2 + 1)-Dimensional sine-Gordon equation; Symmetry group; Soliton

1. Introduction

In nonlinear science, soliton theory plays an essential role and has been widely applied in almost all the natural sciences, especially in all the physics branches such as fluid mechanics, plasma physics, nonlinear optics, condensed matter physics, etc. [1–4]. In particular, some localized coherent theories of higher-dimensional soliton systems have been proved to be of considerable interest. For instance, the discovery, via the Bäcklund transformation by Boiti et al. [5,6], of dromion type coherent solutions of the Davey-Stewartson I system provided renewed interest in higher-dimensional soliton systems. In the past several years, in the study of higher-dimensional nonlinear physical models, much effort has been focused on the propagating solitons [3,7–9]. However, our world is colorful, there may exist many non-propagating solitons, just as Wu et al. reported about the non-propagating hydrodynamical solitons in their experiment [10]. Even though there have been many studies done on non-propagating hydrodynamical solitons both in theoretical and experimental aspects [11,12], to the best of our knowledge, the non-propagating solitons in higher-dimensional nonlinear physical models, especially, in some well-known nonlinear dynamic systems were not much reported in the previous literature.

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On the other hand, we know that the symmetry study plays a very important role in almost all scientific fields, especially in the soliton theory because of the existence of infinitely many symmetries for integrable systems. The classical and the non-classical Lie group approaches are two famous methods of deriving symmetries and conditional symmetries of differential equations. In 1974, Bluman and Cole [13] and later Olver and Rosenau [14] generalized the Lie approach to encompass symmetry transformations so that the invariants become a subset of the possible solutions of a partial differential equation (PDE). Later, a new, say, direct method was developed by Clarkson and Kruskal (CK) [15] in 1989 to obtain the similarity reductions of a PDE without using group theory. This fact hints that there exists a simple method to find symmetry groups for many types of nonlinear equations.

In this paper, the $(2+1)$-dimensional sine-Gordon equation is discussed by using a direct method to get both its Lie point symmetry group, the related (infinitesimal) symmetry algebra and then the exact solutions. From which, two couples of non-propagating dromion-lattice solitons are derived by choosing several appropriate functions.

2. Discussion and results

A $(2+1)$-dimensional master soliton system had been constructed by Konopelchenko and Rogers [16] in 1993 via a reinterpretation, and Loewner generalized a class of infinitesimal Bäcklund transformations in a gas dynamics context. A particular reduction leads to a symmetric integrable extension of the classical sine-Gordon equation, namely,

$$\left(\frac{\phi_x}{\sin\theta}\right)_x - \left(\frac{\phi_y}{\sin\theta}\right)_y + \frac{\phi_y\theta_x - \phi_x\theta_y}{\sin^2\theta} = 0, \tag{1}$$

$$\left(\frac{\phi'_x}{\sin\theta}\right)_x - \left(\frac{\phi'_y}{\sin\theta}\right)_y + \frac{\phi'_y\theta_x - \phi'_x\theta_y}{\sin^2\theta} = 0, \tag{2}$$

where $\theta_t = \phi + \phi'$. After that, the model has been widely studied by many authors [17–20]. In [21], the similarity reductions of (1) and (2) have been discussed. In [22], an equivalent group analysis for a gauge equivalent form of the system had also been given.

The $(2+1)$-dimensional sine-Gordon (2DsG) system (1) and (2) is generated as the compatibility condition of the particular Loewner Konopelchenko Rogers (LKR) triad [16]

$$\left[I\partial_x + \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\partial_y\right]\Phi = 0,$$

$$\left[I\partial_t\partial_y + \frac{1}{2}\begin{pmatrix} 0 & \theta_t \\ -\theta_t & 0 \end{pmatrix}\partial_y - \frac{1}{2\sin\theta}\begin{pmatrix} \phi_y\cos\theta - \phi_x & -\phi'_y\sin\theta \\ \phi_y\sin\theta & \phi'_y + \phi'_x\cos\theta \end{pmatrix}\right]\Phi = 0,$$

$$\left[I\partial_t\partial_x + \frac{1}{2}\begin{pmatrix} 0 & \theta_t \\ -\theta_t & 0 \end{pmatrix}\partial_x - \frac{1}{2\sin\theta}\begin{pmatrix} \phi_x\cos\theta - \phi_y & -\phi'_x\sin\theta \\ \phi_x\sin\theta & \phi'_x + \phi'_y\cos\theta \end{pmatrix}\right]\Phi = 0.$$

It is also known that the above 2DsG equation is equivalent to the following compact version [20]:

$$u_{\xi\eta t} + u_\eta v_{\xi t} + u_\xi v_{\eta t} = 0, \tag{3}$$

$$v_{\xi\eta} = u_\xi u_\eta, \tag{4}$$

where

$$\xi = \frac{1}{2}(y-x), \quad \eta = \frac{1}{2}(y+x), \quad u = \frac{1}{2}\theta$$

and v is determined by

$$v_{\xi t} = \frac{\phi'_\eta - \phi_\eta - \theta_{\eta t}\cos\theta}{2\sin\theta}, \quad v_{\eta t} = \frac{\phi_\xi - \phi'_\xi - \theta_{\xi t}\cos\theta}{2\sin\theta}.$$