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Finite element approximation for fourth-order nonlinear problem in the plane

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Abstract

In this paper, a class of fourth-order nonlinear elliptic problem is investigated in a bounded convex domain $\Omega \subset R^2$. Under some assumptions, the existence and uniqueness of solution are proved via the Schaefer's Fixed Point Theorem. Furthermore, conforming finite element approximation is applied and H^2-error estimate and L^2-error estimate are obtained. Finally, the numerical experiments are provided to verify our theoretical analysis.

Keywords: Fourth-order nonlinear problem; Finite element approximation; Error estimates

1. Introduction

The fourth-order elliptic problem is an important mathematical model in fluid mechanics [1], solid mechanics and other continuous physical system [2]. For example, in fluid mechanics, the stream function ψ of the incompressible flows in R^2 satisfies the biharmonic problem; in solid mechanics, plate bending model is satisfied by harmonic problem, etc. Hence, large numbers of papers appear about this problem. As well known, the well-posedness theory of fourth-order linear problem is similar with second-order-problem [3].

Although many techniques like the maximum principle and comparison principle are not available for fourth-order problem with Dirichlet boundary, others still are valid, such as the methods of upper and lower solutions, and minimax method, in particular, if the equation is a Euler–Lagrange equation associated with some energy functional [4–6]. In addition, some fixed point theorems also are power tools for dealing with the nonlinear problem. We refer the reader to [7,8] for more details.

In this paper, we will study a class of fourth-order nonlinear problem:

$$\Delta a(x, \nabla u, \Delta u) = \phi(|\nabla u|) + f$$

which comes from the mathematical model of the optimal shape design for blade's surface of an impeller, which is a coupled system with nonlinear fourth-order elliptic problem and rotating Navier–Stokes equations

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[9]. Under some assumptions, the Schaefer's Fixed Point Theorem is applied to prove the existence of solution. Comparing with other fixed point theorems, the advantage of Contracting Mapping Principle is that the solution not only exists but also is unique. Hence, in terms of Contracting Mapping Principle, we show that the solution is locally unique.

In numerical analysis, the popular method for fourth-order problem is mixed finite element methods, such as [10–16] and references therein. Moreover, to our best knowledge, [16,17] are only two papers for fourth-order quasilinear problem. In [17], Karchevskii, Lyashko and Timerbaev discussed conforming finite element. But very strong assumptions were required. They assumed that the nonlinear operator of the problem was strongly monotone and Lipschitz continuous, from which the existence and uniqueness of weak solution and approximation solution were immediately obtained in an appropriate weighted Sobolev space. In this paper, we also investigate the finite element approximation using conforming element under some weak assumptions. It is well known that for linear biharmonic problem following error estimates hold (cf. [18]):

$$\begin{cases} \|u-u_h\|_{H^2} \leqslant ch(\|u\|_{H^3}+h\|u\|_{H^4}), \\ \|u-u_h\|_{L^2} \leqslant ch^3(\|u\|_{H^3}+h\|u\|_{H^4}), \end{cases}$$

where $c>0$ is independent of h. In this paper, the H^2-error estimate is showed to be optimal via a linearized technique which is introduced and used to deal with the second-order nonlinear problem in [19]. However, the L^2-error estimate is not optimal unless $a_{y_i}(x,y,z)\equiv 0$, where $y\in R^2, z\in R, i=1,2$.

This paper is organized as follows: in Section 2, we give some preliminary knowledge; in Section 3, the existence and uniqueness of solution are showed; in Section 4, the finite element subspace and the discretized variational formulation are given. Like Section 3, the approximation solution is locally unique; in Section 5, the H^2-error estimate and L^2-error estimate for sufficiently small h are obtained; in Section 6, the numerical experiments are provided to verify our theoretical analysis.

2. Preliminary

Consider a class of fourth-order nonlinear problem:

$$\begin{cases} \Delta a(x,\nabla u,\Delta u) = \phi(|\nabla u|)+f & \text{in } \Omega, \\ u=\frac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $u(x):\Omega\to R$ is unknown, $a(x,y,z):\Omega\times R^2\times R\to R$ is a smooth function. Moreover, $a(\cdot,y,z)$ is nonlinear with respect to y and is linear with respect to z, $\phi(\cdot):R\to R$ is a nonlinear operator, f is independent of u. $\Omega\subset R^2$ is a bounded convex domain with boundary $\partial\Omega$, n denotes the unite outer normal vector to $\partial\Omega$.

Introduce the linear space

$$V=H_0^2(\Omega)=\left\{v\in H^2(\Omega), v=\frac{\partial v}{\partial n}=0 \text{ on } \partial\Omega\right\}$$

with the inner product

$$A(u,v)=\int_\Omega \Delta u\Delta v\,\mathrm{d}x \quad \forall u,v\in V$$

and the norm

$$\|v\|_V = A(v,v)^{\frac{1}{2}} \quad \forall v\in V.$$

Following Sobolev inequalities are usually used:

$$\|u\|_{L^\infty}\leqslant c_1\|u\|_V, \quad \|\nabla u\|_{L^q}\leqslant c_2\|u\|_V \quad \text{for all } 2\leqslant q<+\infty,\ \forall u\in V,$$

where c_1 and c_2 depend only on Ω. Throughout this paper, we always assume the embedding constant $c_1=c_2=1$.