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Applied Mathematics and Computation 197 (2008) 755–759

APPLIED MATHEMATICS AND COMPUTATION

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New iterative methods for non-linear equations

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Abstract

In this paper, some new iterative methods for solving non-linear equations $f(x) = 0$ are presented and analyzed by means of a new two-step predictor–corrector type iterative method. The convergence criteria for these iterative methods are also discussed. Several numerical examples are given to illustrate the efficiency and performance of the new methods.

Keywords: Iterative method; Convergence; Taylor expansion

1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. Newton's method is the most important iterative method for solving non-linear equations. Newton's method for a single non-linear equation is written as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. This is a basic method, which converges quadratically. So far, the techniques of iterative methods for finding the approximate solutions consist of Taylor series, quadrature formulas, homotopy and decomposition (see [2–7] and the reference therein). Most of them can be viewed as alterative method of Newton and its variant forms. Noor [1] mainly discussed some new iterative methods including variant forms by using the variational iteration technique and obtain the different iterative methods through making different variations. In this paper we introduce a new two-step predictor–corrector type iterative method and analyze some new iterative methods by using Taylor series methods. The fact that these new methods are all fourth-order convergent is proved, and the better convergence criteria are obtained by the variation of parameters. At last, several examples are given to illustrate the efficiency and performance of these new methods and through take the different parameter, we can obtain the less number of iterations. These new methods can be considered as a further improvement on Noor [1].

2. Iterative methods

Considering the non-linear equation of the type $f(x) = 0$, assuming that x^* is a simple root and x_0 is an initial value close to x^* sufficiently, we obtain the following iterative method:

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.08.012

Algorithm 2.1. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes, where α is a parameter.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n) - \alpha f(y_n)} \tag{2}$$

Algorithm 2.2. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes, where α is a parameter.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)f'(y_n)}{[f'(y_n)]^2 - \alpha f(y_n)f''(y_n)} \end{aligned} \tag{3}$$

Algorithm 2.3. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes, where α is a parameter.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n) - \alpha f(y_n) + \alpha\left[\frac{f(y_n)}{f'(y_n)}\right]^2 f''(y_n)} \end{aligned} \tag{4}$$

Algorithm 2.4. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes, where α is a parameter.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n) - \alpha f(y_n) - \left[\frac{f(y_n)}{f'(y_n)}\right] f''(y_n)} \end{aligned} \tag{5}$$

The convergence criteria for Algorithms 2.1–2.4 are discussed in the following theorems.

Theorem 2.1. *Assume that the function $f{:}D \subset R \to R$ for an open interval D has a simple root $x^* \in D$. Let $f(x)$ be smooth sufficiently in some neighborhood of the root x^*, and then the method defined by Algorithm 2.1 is fourth-order convergent.*

Proof. Let x^* be a simple root of f. Since f is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about x^*, we get

$$f(x_n) = f'(x^*)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + \cdots) \tag{6}$$

$$f'(x_n) = f'(x^*)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + \cdots) \tag{7}$$

where $e_n = x_n - x^*$, $c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$ and $k = 2, 3, \ldots$

By (6) and (7), we have

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} = {}& e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + (8c_2^4 - 20c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ & + (13c_2c_5 - 28c_2^2c_4 - 5c_6 - 16c_2^5 + 52c_2^3c_3 + 17c_3c_4 - 33c_2c_3^2)e_n^6 + \cdots. \end{aligned} \tag{8}$$

From (1) and (8), we have

$$\begin{aligned} y_n = {}& x^* + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - (8c_2^4 - 20c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ & - (13c_2c_5 - 28c_2^2c_4 - 5c_6 - 16c_2^5 + 52c_2^3c_3 + 17c_3c_4 - 33c_2c_3^2)e_n^6 + \cdots. \end{aligned} \tag{9}$$