Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 197 (2008) 891–903
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Heatlet approach to diffusion equation on unbounded domains

T. Gnana Bhaskar [a,*], S. Hariharan [a], Neela Nataraj [b]

[a] Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
[b] Department of Mathematics, Indian Institute of Technology, Mumbai 400 076, India

Abstract

We develop Heatlets, the fundamental solutions of heat equation using wavelets, for numerically solving inhomogeneous and homogeneous initial value problems of diffusion equation on unbounded domains. Unlike finite difference and finite element methods, diffusion into an infinite medium is satisfied analytically, avoiding the need for artificial boundary conditions on a finite computational domain. The approach is applied to a number of examples and the numerical results confirm the theoretical findings.

Keywords: Heatlets; Artificial boundary conditions; Wavelets; Heat equation

1. Introduction

The study of heat equation on unbounded domains arises in the modeling of a variety of physical and engineering applications. Finding the numerical solution to such problems is a non-standard task. Employing the finite difference or finite element methods requires the use of artificial boundary conditions (ABC's) imposed on the finite computational domain to simulate the effects of diffusion into an infinite medium. There is an extensive literature on the issues concerning the treatment of ABC's and here we only mention the monograph [4] and the works of Han and Huang [7] and Wu and Sun [13]. It is well known that the adhoc discretization of the analytic ABC's induces numerical reflections at the artificial boundary and the stability properties of the underlying method could also be affected. To avoid such difficulties, the discrete ABC's are obtained directly from the fully discretized problem on the unbounded domain by [3] and have been successfully employed in [2] and several others in a variety of applications.

On the other hand, integral equation methods have also been employed for finding the numerical solution of these problems, albeit with a high computational cost. Recently Greengard and Lin [5] developed an efficient new algorithm based on the spectral approximation of the free space heat kernel and the non-uniform fast Fourier transform. For more details on this approach, we refer to [6,10] and the references therein.

* Corresponding author.
E-mail addresses: gtenali@fit.edu (T. Gnana Bhaskar), hseethar@fit.edu (S. Hariharan), neela@math.iitb.ac.in (N. Nataraj).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.08.060

The method proposed in this paper uses wavelets as building blocks to generate certain fundamental solutions, called heatlets, to the heat equation. In spite of the popularity of wavelets and their use in numerical schemes for finding approximate solutions to problems arising in physical and engineering applications, the theoretical connections between wavelets and differential operator have not been thoroughly investigated. Shen and Strang [9] made some pioneering contributions in this direction. More precisely, heatlets are fundamental solutions of the problem

$$u_t = u_{xx}, \qquad u(x,0) = f(x), \quad -\infty < x < \infty, t > 0, \tag{1.1}$$

where the initial function $f(x)$ is either a scaling function or a wavelet. Establishing a theoretical alliance between wavelets and heat equation, they studied heatlets for their translation and scale invariance properties. The locality and their vanishing moment properties are studied by Shen [8]. However, to the best of our knowledge, no numerical study has been done using these heatlets. In this paper, we introduce a forcing term in (1.1), which may be either a scaling function or a wavelet and obtain the corresponding heatlet solutions. Further, we propose here a method for finding the numerical solution of the heat conduction problems on unbounded domains using heatlets. We compare the numerical results with those obtained using finite difference and finite element methods.

The main advantage of the method proposed here is that once the library of heatlets are built for a heat equation up to a desired level of accuracy, the computation of solution for any initial function or forcing term, requires only the knowledge of wavelet coefficients of these functions and the numerical solution of the problem can be easily obtained.

The organization of the paper is as follows. In Section 2, we provide some mathematical preliminaries on wavelets. Section 3 deals with the construction of heatlets, their properties and a description of the proposed numerical approach. In Section 4, we present briefly the problem on the computational domain, along with the ABC's. Finally, numerical examples are considered in Section 5 and the results obtained are compared with those obtained using the finite difference and finite element methods.

2. Preliminaries

A multiresolution analysis (MRA) of $L^2(\mathbb{R})$, the real space of all square integrable functions on $\mathbb{R}$, equipped with the standard innerproduct $(\cdot,\cdot)$, is a chain of closed subspaces indexed by all integers:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots, \tag{2.1}$$

such that

(i) $\overline{\lim_{n\to+\infty} V_n} = L^2(\mathbb{R})$,
(ii) $\overline{\lim_{n\to-\infty} V_n} = \{0\}$,
(iii) $f(\cdot) \in V_n \iff f(2\cdot) \in V_{n+1}$.
(iv) Let ϕ be a scaling function such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ constitute a complete orthonormal basis of V_0.

To obtain a multiresolution analysis, it suffices to construct the scaling function $\phi(x)$. The entire subspace chain can then be reconstructed from $\phi(x)$ according to (iii) and (iv). Since $V_0 \subset V_1$ and from (iii) and (iv), it is easy to see that $\phi(\cdot)$ must be a linear combination of $\{\phi(2\cdot -k) : k \in \mathbb{Z}\}$, leading to the two scale relation

$$\phi(\cdot) = 2\sum_{k\in\mathbb{Z}} h_k\phi(2\cdot -k), \tag{2.2}$$

for a suitable set of coefficients $(\dots, h_{-1}, h_0, h_1, \dots)$. If the scaling function ϕ is compactly supported, as is usually the case in most numerical evaluations, we have

$$\phi(\cdot) = 2\sum_{k=0}^{L} h_k\phi(2\cdot -k),$$

with $h_0h_L \neq 0$. It is usually assumed in wavelet analysis that $\int \phi = 1$. This implies

$$h_0 + h_1 + h_2 + \dots + h_L = 1. \tag{2.3}$$