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Non-simultaneous quenching in a semilinear parabolic system with weak singularities of logarithmic type

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Abstract

In this paper, we are interested in the possibility of non-simultaneous quenching for positive solutions of a coupled system of two semilinear parabolic equations with weak singularities of logarithmic type, $u_t = u_{xx} + \log(\alpha v)$, $v_t = v_{xx} + \log(\beta u)$, $0 < \alpha, \beta < 1$, with homogeneous Neumann boundary conditions and positive initial data. Under some assumptions on the initial data and parameters α, β, we prove that the quenching is always non-simultaneous. We also give the quenching rate when the quenching is non-simultaneous. Finally, we show that our results can be used to a blow-up problem.

Keywords: Weak singularity; Quenching; Quenching point; Quenching rate; Semilinear parabolic system

1. Introduction

In this paper, we study the solutions of the following semilinear parabolic system

$$\begin{aligned} u_t &= u_{xx} + \log(\alpha v), \\ v_t &= v_{xx} + \log(\beta u), \end{aligned} \quad \text{in } (0,1)\times(0,T) \tag{1.1}$$

with Neumann boundary conditions

$$\begin{aligned} u_x(0,t) &= v_x(0,t) = 0, \\ u_x(1,t) &= v_x(1,t) = 0, \end{aligned} \quad \text{in } (0,T) \tag{1.2}$$

and initial data

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x), \end{aligned} \quad \text{in } [0,1]. \tag{1.3}$$

Throughout this paper, we assume that $0 < \alpha, \beta < 1$, and $u_0(x)$, $v_0(x)$ are smooth enough and satisfy $0 < u_0(x)$, $v_0(x) \leqslant 1$ for all $x \in [0,1]$, and compatible with the boundary data.

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We are interested in the quenching phenomenon of the solution, which can be formulated as follows, see for instance, [12,1,5,11,9], and some survey papers [3,10,8]. We say that the solution $(u(x,t), v(x,t))$ of the problem (1.1)–(1.3) quenches, if (u, v) exists in the classical sense and is positive for all $0 \leqslant t < T$ and satisfies

$$\inf_{t\uparrow T} \min_{0\leqslant x\leqslant 1} \{u(x,t), v(x,t)\} = 0.$$

If this happens, T will be called as quenching time. Clearly at quenching time T, a singularity develops in the absorption term, consequently the classical solution can no longer exists.

There are many papers considering the quenching phenomenon for the solutions of nonlinear parabolic systems. In [12], de Pablo etc. considered the following problem

$$\begin{aligned} &u_t = u_{xx} - v^{-p}, \\ &v_t = v_{xx} - u^{-q}, \end{aligned} \quad \text{in } (0,1)\times(0,T),$$

with the same initial-boundary conditions as (1.2), (1.3). Their results show that if $p, q > 1$, then quenching is always simultaneous, if $p < 1$ or $q < 1$, then there exists a wide class of initial data with *non-simultaneous quenching* (i. e. one of the components of the system remains bounded away from zero, while the other vanishes at some point at time T), and if $p < 1 \leqslant q$ or $q < 1 \leqslant p$, then quenching is always non-simultaneous. They also gave the quenching rate in all cases, especially if quenching is non-simultaneous and, for instance, v is the quenching variable, then for t close to T,

$$v(0,t) \sim (T - t). \tag{1.4}$$

Here as usual, the notation $f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1 g \leqslant f \leqslant c_2 g$.

It is well-known that the singularity of absorption term $\log(\alpha v)$ (respectively, $\log(\beta u)$) in (1.1) is weaker than that of $-v^{-p}$ (respectively, $-u^{-q}$), see [14,13] for detail. Therefore, we are interested in whether the weakness of singularity can lead to the non-simultaneous quenching, and whether the solution of (1.1)–(1.3) still admits the same quenching rate estimate as (1.4).

As far as the single semilinear parabolic equation with singularity of logarithmic type is concerned, in [13], Salin considered the problem

$$\begin{aligned} &u_t = u_{xx} + \log(\alpha u), \quad \text{in } (-l,l)\times(0,T), \\ &u(x,0) = u_0(x), \quad x \in [-l,l], \\ &u(\pm l,t) = 1, \quad t \in [0,T), \end{aligned}$$

and derived quenching rate

$$\lim_{t\uparrow T} \left(1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{\mathrm{d}s}{\log(\alpha s)}\right) = 0$$

uniformly, when $|x| < C\sqrt{T-t}$, for every $C \in (0,\infty)$. In [15], we studied the quenching behavior for

$$\begin{aligned} &u_t = u_{xx} + \epsilon\|u(\cdot,t)\|^q \log(\alpha u), \quad 0<x<1, \quad t>0, \\ &u(0,t) = u(1,t) = 1, \quad t>0, \\ &u(x,0) = u_0(x), \quad x \in [0,1], \end{aligned} \tag{1.5}$$

where $\|u(\cdot,t)\| := \int_0^1 u(x,t)\,\mathrm{d}x$, $\epsilon > 0$, $q > 0$, $0 < \alpha < 1$, $u_0(0) = u_0(1) = 1$, and got the quenching rate

$$\lim_{t\uparrow T} \left(1 + \frac{(\epsilon\sigma^q)^{-1}}{T-t}\int_0^{u(x,t)} \frac{\mathrm{d}\eta}{\log(\alpha\eta)}\right) = 0 \tag{1.6}$$

uniformly for $|x - \frac{1}{2}| < C\sqrt{T-t}$, where $C > 0$ is an arbitrary constant, and $\sigma = \int_0^1 u(x,T)\,\mathrm{d}x$. From these two results we see that the formulation of quenching rate is quite the same no matter what the weak singular term would be(either local or non-local). But what it would be if coupled weak singular terms appear for the same type of semilinear parabolic system? Our result of this paper for the case of non-simultaneous quenching shows that the formulation of the quenching rate is quite different from those of the above two, in fact, it