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A nonconforming covolume method for elliptic problems

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Abstract

We consider a control volume (covolume) method for second-order elliptic PDEs with the rotated-Q_1 nonconforming finite element on rectangular grids. The coefficient κ may a variable, diagonal tensor, or discontinuous. We prove first-order convergence in H^1 norm and second order convergence in L^2 norm when the partition is square. Our numerical experiments show that our covolume scheme has about 30% less error than FEM even when κ is discontinuous tensor.

Keywords: Covolume method; Rotated bilinear finite element method; Duality; Optimal order

1. Introduction

In this paper, we consider a covolume scheme with Q_1 nonconforming finite element method (FEM) for the second-order elliptic equations with tensor coefficients. We prove first-order convergence in H^1 norm and second-order convergence in L^2 norm. This scheme was first introduced for Stokes problems in [3,6]. We apply it to elliptic problems with variable, discontinuous, and tensor coefficient. For other type of finite volume methods, we refer to [1,2,4–7,11–13,15] and for finite element method, we refer to [8,9,14].

The analysis of these covolume methods can be well described if we introduce a transfer operator γ from usual FEM space to the space of piecewise constant. As a result, the scheme can be viewed as a Galerkin scheme rather than a Petrov–Galerkin scheme.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following second-order elliptic boundary value problem:

$$\begin{aligned} -\mathrm{div}(\kappa\nabla u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\kappa = \kappa(x) := \mathrm{diag}(\kappa_1(x), \kappa_2(x))$ is a diagonal and uniformly positive definite matrix. For Q_1 nonconforming finite element method, we only consider the case where Ω is a union of axi-parallel domain. Let $h > 0$ be a

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parameter and let $\mathcal{T}_h = \{K\}$ be a regular family of partition into rectangles of Ω in the sense that there exist constants $c > 0$, $C > 0$ such that

$$ch^2 \leqslant |K| \leqslant Ch^2 \quad \text{for all} \quad K \in \mathcal{T}_h.$$

Now we describe a Q_1 nonconforming finite element method. Let $S_K = \{v_h = a + bx_1 + cx_2 + d(x_1^2 - x_2^2) \text{ on } K\}$ and let

$$N_h = \left\{ v_h|_K \in S_K, \int_{K\cap\partial K} v_h \, \mathrm{d}\sigma = \int_{K'\cap\partial K'} v_h \, \mathrm{d}\sigma \quad \text{if } K, K' \text{are adjacent, and } \int_{\partial K\cap\partial\Omega} v_h \, \mathrm{d}\sigma = 0 \right\}. \tag{2}$$

The usual nonconforming variational formulation of (1) is defined through element-wise form, i.e, we let

$$a_h(v_h, w_h) = \sum_K (\kappa\nabla v_h, \nabla w_h)_K, \quad \text{for } v_h, w_h \in N_h.$$

Let $\|\cdot\|_{0,D}$ (dropping D when $D = \Omega$) denote the usual $L^2(D)$ norm on a domain D; $D = K$ or Ω, ∂K, $\partial\Omega$, etc., and $|\cdot|_{1,h} = |a_h(\cdot,\cdot)|^{1/2}$ be the discrete energy norm induced by $a_h(\cdot,\cdot)$. Then it is well-known [14,4,12,8] that the nonconforming finite element solution defined by

$$a_h(\tilde{u}_h, v_h) = (f, v_h), \quad \text{for all } v_h \in N_h,$$

satisfies

$$\|u - \tilde{u}_h\|_0 + h|u - \tilde{u}_h|_1 \leqslant Ch^2\|f\|_0. \tag{3}$$

Now we consider a covolume formulation of the problem (1). For that purpose, we need to subdivide the given partition by connecting the vertices of each element with its center C, resulting in four subtriangles. Now the region consisting of two adjacent triangles sharing a common edge is denoted by K^*, called a covolume. The midpoints of edges are denoted by $m_i, i = 1, \ldots, 4$. To define the covolume method, we need another space (Fig. 1)

$$W_h = \{v_h|_{K^*} \quad \text{is constant on each } K^*, \text{ and 0 on the boundary covolumes}\} \tag{4}$$

and an operator connecting N_h to W_h. Let K_j^* be the covolume with associated edge e_j, and m_j denote its mid point. Let $\bar{v}_h = \frac{1}{|e_j|}\int_{e_j} v_h d\sigma$ be the average of v_h on the edge e_j. Then we introduce a transfer operator $\gamma : N_h \to W_h$ by

$$\gamma v_h|_{K^*} = \sum_{K_j^*} \bar{v}_h(m_j)\chi_j(x),$$

where χ_j is the characteristic function of K_j^*.

Find $u_h^* \in N_h$ such that

$$a_h^*(u_h^*, v_h) = (f, \gamma v_h) \quad \text{for all } v_h \in N_h, \tag{5}$$

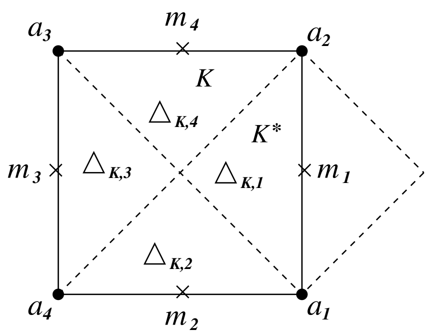

Fig. 1. A typical element K and its covolume partition K^*.