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Extremal solutions for nonresonance impulsive functional dynamic equations on time scales ☆

John R. Graef [a,*], Abdelghani Ouahab [b]

[a] *Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, United States*
[b] *Department of Mathematics, University of Sidi Bel Abbes, BP 89 2000 Sidi Bel Abbes, Algeria*

Abstract

In this paper, we give an existence theorem for extremal solutions of a class of first order nonresonance impulsive dynamic equations on time scales with a periodicity condition.

Keywords: Delta derivative; Extremal solutions; Impulsive functional dynamic equations; Fixed point; Nonresonance; Time scales

1. Introduction

In this paper, we study the existence of extremal solutions of the nonresonance impulsive functional dynamic equation on a time scale

$$y^{\Delta}(t) - \lambda y^{\sigma}(t) = f(t, y_t), \quad \text{a.e. } t \in J := [0, b] \cap \mathbb{T}, \ t \neq t_k, \ k = 1, \dots, m, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \ y(0) = y(b), \tag{3}$$

where Δ denotes the delta derivative defined below, $\mathbb{T}$ is time scale, $\lambda \in \mathbb{R}_+$, $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a given function,

$$\mathscr{D} = \{\psi : [-r, 0] \to \mathbb{R} | \psi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}^-) \text{ and } \psi(\bar{t}^+) \text{ exist, and } \psi(\bar{t}^-) = \psi(\bar{t})\},$$

$\phi \in \mathscr{D}$, $0 < r < \infty$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in \mathbb{T}$, $0 = t_0 < t_1 < \dots t_m < b$, and

$$y_t(\theta) = y(t + \theta) \quad \text{for } \theta \in [-r, 0] \text{ and } t \in J - \{t_1, t_2, \dots, t_m\}.$$

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* Corresponding author.
E-mail addresses: john-graef@utc.edu (J.R. Graef), ouahab@univ-sba.dz (A. Ouahab).

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Here, $y(t_k^+)$ and $y(t_k^-)$ represent the right and left hand limits of $y(t)$ at $t = t_k$ in the time scale sense, respectively. Moreover, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, whereas, if t_k is left scattered, then $y(t_k^-) = y(t_k)$. The function σ, known as the forward jump operator, will be defined below, and $y^\sigma(t) = y(\sigma(t))$.

In recent years, impulsive differential equations have gained a lot of attention due to their importance as mathematical models of phenomena in physics, chemistry, population dynamics, biotechnology, economics, and many other areas. Basic results on impulsive equations can be found in the works of Bainov and Simeonov [6], Lakshmikantham et al. [21], and Samoilenko and Perestyuk [24]. In the case where $\mathbb{T} = \mathbb{R}$, nonresonance problems for impulsive differential equations were studied, for example, by Benchohra and Eloe [8] and Nieto [23].

For the basic theory of dynamic equations on times scales and some of their applications, we refer the reader to the monographs by Bohner and Peterson [12,13] and Lakshmikantham et al. [22] and to the papers by Agarwal and Bohner [2], Agarwal et al. [3], Bohner and Eloe [11], Bohner and Peterson [14], Erbe and Peterson [15,16], and the references contained therein. The existence of solutions of boundary value problems on time scales was recently studied by Agarwal and O'Regan [4], Anderson [5], Henderson and Tisdell [20], and Sun [25], and impulsive dynamic equations on time scales were examined recently by Agarwal et al. [1], Benchohra et al. [9,10], Graef and Ouahab [17], and Henderson [19]. The study of extremal solutions of first order dynamic equations on time scales was initiated by Belarbi et al. [7].

We will prove our existence theorem for the problem (1)–(3) by using the Heikkila and Lakshmikantham fixed point theorem [18] on the existence of the least and the greatest fixed point for an operator defined on an ordered interval.

2. Preliminaries

We begin by recalling some concepts from the time scales calculus that we will utilized in the sequel. A *time scale* $\mathbb{T}$ is an nonempty closed subset of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively, (supplemented by $\inf \emptyset := \sup \mathrm{T}$ and $\sup\emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is *left-dense*, *left-scattered*, *right-dense*, or *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, we define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The symbols $[a, b]$, $[a, b)$, etc., denote time scales intervals, e.g.,

$$[a, b] = \{t \in \mathbb{T} : a \leqslant t \leqslant b\},$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 2.1. Let X be a Banach space. The function $f : \mathbb{T} \to X$ is *rd*-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point; in this case, we will write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, X)$.

Definition 2.2. For $t \in \mathbb{T}^k$, the Δ derivative of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) such that for every $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

Remark 2.1. It is known that if f is continuous, then it is *rd*-continuous. Moreover, if f is delta differentiable at t, then it is continuous there.

Definition 2.3. The function F is an antiderivative of $f : \mathbb{T} \to X$ provided

$$F^\Delta(t) = f(t) \quad \text{for each} \quad t \in \mathbb{T}^k.$$