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# Approximating fixed points of nonexpansive mappings in a Banach space by metric projections

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**Abstract**

In this paper, a strong convergence theorem for nonexpansive mappings in a uniformly convex and smooth Banach space is proved by using metric projections. This theorem is different from the recent strong convergence theorem due to Xu [H.K. Xu, Strong convergence of approximating fixed point sequences for nonexpansive mappings, Bull. Aust. Math. Soc. 74 (2006) 143–151] which was established by generalized projections.


*Keywords:* Nonexpansive mapping; Metric projection; Uniformly convex Banach space; Approximating fixed point

## 1. Introduction

Let $C$ be a closed convex subset of a real Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping. Strong convergence theorems for nonexpansive mappings have been investigated with implicit and explicit iterative schemes; see Browder [2], Halpern [5], Reich [9], Takahashi and Ueda [13], Wittmann [14] and Shioji and Takahashi [10] etc. On the other hand, using the metric projection, Nakajo and Takahashi [7] introduced the following iterative algorithm in the framework of Hilbert spaces: $x_0 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0, \alpha], \alpha \in [0, 1)$ and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space $H$ onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of $T$. The authors [6] extended Nakajo and Takahashi's theorem to Banach spaces by using relatively nonexpansive mappings.

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Xu [15] recently introduced the following iterative algorithm in the framework of Banach spaces: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - Tz\| \leqslant t_n\|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} = \Pi_{C_n \cap D_n}x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.2}$$

where $\overline{co}D$ denotes the convex closure of the set $D$, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $\Pi_{C_n \cap D_n}$ is the generalized projection of $E$ onto $C_n \cap D_n$ (see Alber [1] for generalized projections). Then, he proved that $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of $T$.

In this paper, motivated by (1.1) and (1.2), we introduce the following iterative algorithm for finding fixed points of nonexpansive mappings in a uniformly convex and smooth Banach space: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - Tz\| \leqslant t_n\|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.3}$$

where $P_{C_n \cap D_n}$ is the metric projection from $E$ onto $C_n \cap D_n$. We first prove that the sequence $\{x_n\}$ generated by (1.3) is well-defined. Then, we prove that $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from $E$ onto the set of all fixed points of $T$.

## 2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let $E$ be a real Banach space and let $E^*$ be the dual of $E$. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^*\rangle$. When $\{x_n\}$ is a sequence in $E$, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The normalized duality mapping $J$ from $E$ to $E^*$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Some properties of the duality mapping have been given in [4,11,12].

A Banach space $E$ is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space $E$ is also said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n + y_n\| = 2$. We also know that if $E$ is a uniformly convex Banach space, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. Then the Banach space $E$ is said to be *smooth* if

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. Let $C$ be a closed convex subset of a reflexive, strictly convex and smooth Banach space $E$. Then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y\in C}\|y - x\|.$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from $E$ onto $C$. Let $x \in E$ and $u \in C$. Then, it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u)\rangle \geqslant 0 \tag{2.1}$$

for all $y \in C$ [1,8,11,12].

Let $C$ be a closed convex subset of a Banach space $E$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed point of $T$. The following proposition was proved by Bruck [3].

**Proposition 2.1** (See [3]). *Let $C$ be a closed convex subset of a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*