Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 196 (2008) 433–447
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

# Observations on the fifth-order WENO method with non-uniform meshes

Rong Wang [a,1], Hui Feng [b,2], Raymond J. Spiteri [a,*,1]

[a] *Department of Computer Science, University of Saskatchewan, Saskatoon, SK, Canada S7N 5C9*
[b] *Department of Mathematics, Wuhan University, Wuhan, Hubei, China*

**Abstract**

The weighted essentially non-oscillatory (WENO) methods are a popular high-order spatial discretization for hyperbolic partial differential equations. Typical treatments of WENO methods assume a uniform mesh. In this paper, we give explicit formulas for the finite-volume, fifth-order WENO (WENO5) method on non-uniform meshes in a way that is amenable to efficient implementation. We then compare the performance of the non-uniform mesh approach with the classical uniform mesh approach for the finite-volume formulation of the WENO5 method. We find that the numerical results significantly favor the non-uniform mesh approach both in terms of computational efficiency as well as memory usage. We expect this investigation to provide a basis for future work on adaptive mesh methods coupled with the finite-volume WENO methods.


*Keywords:* Finite-volume discretization; Fifth-order WENO method; Non-uniform mesh

## 1. Introduction

The essentially non-oscillatory (ENO) methods were first introduced by Harten et al. in [1,2]. They were the first successful high-order methods for the spatial discretization of hyperbolic conservation laws that had the ENO property. This property is considered to be very useful in the numerical solution of hyperbolic conservation laws because numerical methods often produce spurious oscillations when applied to such problems, especially near shocks or other discontinuous behavior of the solution. The finite-volume ENO spatial discretization was studied in [2], where it was shown to have uniform high-order accuracy right up to the location of any discontinuities. Later Shu and Osher [3,4] developed the finite-difference ENO method. The main idea behind ENO methods is to choose from among several candidates the stencil on which the solution varies

* Corresponding author.
*E-mail addresses:* rong@cs.usask.ca (R. Wang), hui_feng163@tom.com (H. Feng), spiteri@cs.usask.ca (R.J. Spiteri).
[1] The work of this author was partially supported by MITACS, NSERC, and Martec, Ltd.
[2] The work of this author was partially supported by MITACS and NSET.

0096-3003/$ - see front matter 
doi:10.1016/j.amc.2007.06.024

the most smoothly and then approximate the flux at the cell boundaries with a high order of accuracy, thus avoiding the large spurious oscillations caused by interpolating data across discontinuities.

Weighted ENO (WENO) methods were developed in [5,6] to address potential numerical instabilities in choosing ENO stencils. WENO methods use a convex combination of all the ENO candidate stencils; i.e., rather than choosing one specific ENO stencil, each stencil is assigned a weight between 0 and 1. Given $r$ ENO stencils of order $r$, the weights for the WENO method are chosen such that the numerical flux is approximated to order $(2r-1)$ in smooth regions, while in regions near solution discontinuities WENO methods emulate ENO methods so that the ENO property is achieved. In other words, WENO methods achieve a higher order of accuracy than ENO methods in smooth regions, while retaining the ENO property at discontinuities.

Explicit formulas for WENO coefficients on uniform meshes appear in, e.g., [7,8]. A framework for deriving WENO coefficients for non-uniform meshes is established in [7]; explicit formulas appear for uniform meshes. Finite-difference WENO schemes with orders from 7 to 13 are derived in [12] for one-dimensional uniform meshes. Schemes of third and fifth order in multiple spatial dimensions on uniform meshes are derived in [6,13]. WENO coefficients on arbitrary triangular meshes are derived for second-order schemes in [14] and for third- and fourth-order schemes in [15]. Computations in two dimensions with WENO discretizations are performed in [8] on triangular and rectangular meshes.

Given a fixed uniform mesh, the finite-difference WENO methods and the finite-volume WENO methods produce identical spatial discretization operators for one-dimensional, linear, constant-coefficient partial differential equations (PDEs). Of course, they do differ in the quantities that they evolve; i.e., the finite-difference approach evolves point values whereas the finite-volume approach evolves cell averages. For a nonlinear scalar hyperbolic PDE, the equivalence of the finite-difference and finite-volume WENO spatial discretization operators does not hold anymore; however, the computational costs for the two methods are still the same.

For multi-dimensional problems, the finite-difference WENO methods are significantly less computationally expensive than the finite-volume WENO methods [7,8]. Specifically, the finite-difference WENO methods are about 4 times less expensive than the finite-volume WENO methods of the same order for two-dimensional problems. This becomes about 9 times less expensive for three-dimensional problems. In this sense the finite-difference WENO methods are more favorable than the finite-volume methods. Furthermore, when the finite-volume WENO methods are applied for multi-dimensional problems, negative weights may arise [8]; specialized techniques have been used to overcome the difficulty [8].

In recent years, adaptive mesh methods have been used with great success for parabolic problems; see, e.g., [9,10]. They have also been used to solve hyperbolic conservation laws, e.g., [11]. We note that the finite-difference WENO method of third order or higher can only be applied to uniform or smoothly varying meshes [7], i.e., a mesh such that a smooth transformation $\xi = \xi(x)$ transforms the original mesh into a uniform mesh in the new variable $\xi$. This eliminates the possibility of using non-uniform or adaptive meshes with finite-difference WENO methods. In this paper, we focus only on one-dimensional problems, where the computational costs of the finite-volume and finite-difference WENO methods are the same, so we restrict our comparison to the relative efficiency of the finite-volume WENO methods on uniform and non-uniform meshes.

At present it is not known whether the finite-volume WENO methods on a non-uniform (adaptive) mesh can compete with the finite-difference WENO methods on a uniform mesh in terms of efficiency. For example, there are many efficiencies afforded to implementations using uniform meshes in terms of being able to precompute coefficients. In this paper, we perform a quantitative comparison of the relative efficiency of non-uniform mesh approach with the uniform mesh approach for the finite-volume, fifth-order WENO (WENO5) method. Our numerical results show that the use of non-uniform meshes can lead to significant improvements in efficiency over the use of uniform meshes, both in terms of computational efficiency as well as memory usage. These are the first such quantitative comparisons to be made available of which we know. This leads us to hypothesize that if a suitable adaptive mesh algorithm can be derived, an adaptive finite-volume WENO approach can generally outperform the classical finite-difference WENO approach on uniform meshes. We leave the development of an adaptive mesh strategy as future work. We hope this investigation will provide a basis for future work on adaptive mesh methods coupled with the finite-volume WENO methods.

The remainder of this paper proceeds as follows. For completeness, in Section 2 we give explicit, detailed formulas for the coefficients of the finite-volume WENO5 method on non-uniform meshes. We note that these