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Equality conditions for lower bounds on the smallest singular value of a bidiagonal matrix

Yusaku Yamamoto

Nagoya University, Department of Computational Science and Engineering, Nagoya, Aichi 464-8603, Japan

Abstract

Several lower bounds have been proposed for the smallest singular value of a square matrix, such as Johnson's bound, Brauer-type bound, Li's bound and Ostrowski-type bound. In this paper, we focus on a bidiagonal matrix and investigate the equality conditions for these bounds. We show that the former three bounds give strict lower bounds if all the bidiagonal elements are non-zero. For the Ostrowski-type bound, we present an easily verifiable necessary and sufficient condition for the equality to hold.

Keywords: Singular values; Lower bounds; Equality conditions; Bidiagonal matrix; dqds Algorithm

1. Introduction

The singular values are fundamental quantities that describe the properties of a given matrix. In particular, the smallest singular value plays a special role in numerical linear algebra and several lower bounds for estimating it from below have been proposed so far. Examples of lower bounds include Johnson's bound [1], Ostrowski-type bound [2], Brauer-type bound [2] and Li's bound [3].

In a certain situation, we are interested to know whether equality holds in these lower bounds. For example, lower bounds can be used to determine the shifts in the dqds or related algorithms for singular value computation [4,5]. In that case, to guarantee global convergence and numerical stability, we must make sure that the bound is strictly smaller than the smallest singular value [6].

In this paper, we focus on a bidiagonal matrix and study the equality conditions for the four lower bounds listed above. We show that if all the diagonal and upper subdiagonal elements are non-zero, Johnson's bound, Brauer-type bound and Li's bound all give strict lower bounds. The restriction here is not serious since any bidiagonal matrix can be transformed easily to satisfy it. For Ostrowski-type bound, we give a necessary and sufficient condition for equality to hold.

E-mail address: yamamoto@na.cse.nagoya-u.ac.jp

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In Section 2, we review the four lower bounds for the smallest singular value. In Section 3, we give two theorems concerning the equality conditions for these bounds. Section 4 gives an example of a bidiagonal matrix for which the Ostrowski-type bound gives the exact smallest singular value.

2. Lower bounds on the smallest singular value

We consider an n by n upper bidiagonal matrix B given by

$$B = \begin{pmatrix} b_{11} & -b_{12} & & \\ & b_{22} & \ddots & \\ & & \ddots & -b_{n-1,n} \\ & & & b_{nn} \end{pmatrix} \tag{1}$$

and denote its smallest singular value by $\sigma_n(B)$. Following [4], we assume that B has Property (A) defined below:

Definition 2.1. An upper bidiagonal matrix B is said to have Property (A) if all the diagonal elements are positive and all the upper subdiagonal elements are negative, i.e., $b_{ii} > 0$ $(i = 1, \ldots, n)$ and $b_{i,i+1} > 0$ $(i = 1, \ldots, n-1)$.

If B has Property (A), the right and left singular vectors corresponding to $\sigma_n(B)$ can be chosen positive since they are the eigenvectors of positive matrices $(B^{\mathrm{T}}B)^{-1}$ and $(BB^{\mathrm{T}})^{-1}$, respectively, corresponding to the largest eigenvalue $(\sigma_n(B))^{-2}$.

We can show that our assumption is not restrictive as follows. If one of the subdiagonal elements of B is zero, B is decomposed into a direct sum of two upper diagonal matrices. So we can compute the lower bounds for each matrix separately. If a diagonal element of B is zero, by applying one step of the dqds algorithm with zero shift, we can chase the zero element to the lower-right corner. By deflating the element, we obtain a smaller bidiagonal matrix with non-zero diagonals [4]. Finally, the diagonal elements and upper subdiagonal elements can be made positive and negative, respectively, by multiplying appropriate diagonal matrices with diagonal elements ± 1 from both sides.

Now we can state the four lower bounds that we will deal with in this paper. In the following, we adopt the convention that $b_{0,1} = b_{n,n+1} = 0$.

Theorem 2.2 (*Johnson bound* [1])

$$\sigma_n(B) \geqslant \min_{1 \leqslant k \leqslant n} \left\{ b_{kk} - \frac{1}{2}(b_{k-1,k} + b_{k,k+1}) \right\}. \tag{2}$$

Theorem 2.3 (*Brauer-type bound* [2])

$$\sigma_n(B) \geqslant \min_{1 \leqslant j < k \leqslant n} \frac{1}{2} \left\{ b_{kk} + b_{jj} - \sqrt{(b_{kk} - b_{jj})^2 + (b_{k-1,k} + b_{k,k+1})(b_{j-1,j} + b_{j,j+1})} \right\}. \tag{3}$$

Theorem 2.4 (*Li's bound* [3])

$$\sigma_n(B) \geqslant \min_{1 \leqslant k \leqslant n-1} \frac{1}{2} \left\{ b_{kk} + b_{k+1,k+1} - \sqrt{(b_{kk} - b_{k+1,k+1})^2 + (b_{k-1,k} + b_{k,k+1})(b_{k,k+1} + b_{k+1,k+2})} \right\}. \tag{4}$$

Theorem 2.5 (*Ostrowski-type bound* [2])

$$\sigma_n(B) \geqslant \min_{1 \leqslant k \leqslant n} \left\{ \sqrt{b_{kk}^2 + \frac{1}{4}(b_{k-1,k} - b_{k,k+1})^2} - \frac{1}{2}(b_{k-1,k} + b_{k,k+1}) \right\}. \tag{5}$$