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Positive eigenvector of nonlinear perturbations of nonsymmetric M-matrix and Newton iterative solution ☆

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Abstract

Positive eigenvector of nonlinear perturbations of nonsymmetric M-matrix and its Newton iterative solution are studied. It is shown that any number greater than the smallest positive eigenvalue of the M-matrix is an eigenvalue of the nonlinear problem and that the corresponding positive eigenvector is unique and the Newton iteration of the positive eigenvector is convergent. Moreover, such positive eigenvectors form a monotone increasing and continuous function of the corresponding eigenvalues.

Keywords: M-Matrix; Nonlinear perturbation; Eigenvalue; Newton iterative solution

1. Introduction

In [1–3], Choi et al. obtained the Perron–Frobenius theorem of nonlinear perturbations of symmetric positive M-matrices. The theorem is used to research numerical solutions of Gross–Pitaevskii equation, which plays a important role in modeling of the Bose–Einstein condensation of matter at near absolute zero temperatures, see for example [5], and references therein. However, we know that not all discretization techniques for Gross–Pitaevskii equation lead to a Stieltjes matrix, but an M-matrix. In this case, we need a theory on nonlinear perturbations of nonsymmetric M-matrices corresponding to the Perron–Frobenius theorem. In this paper, we study this problem and its Newton iterative solution.

We consider the following nonlinear eigenvalue problem:

$$Ax + F(x) = \lambda x, \tag{1}$$

where A is an $n \times n$ nonsingsular M-matrix, the function $F(x)$ is diagonal, that is

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$$F(x) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix}$$

with the property that $f_i(x_i) > 0$ if $x_i > 0$.

Our objectives are to characterize (componentwise) positive solution x of Eq. (1), which are called the positive eigenvector of nonlinear operator $G(x) = Ax + F(x)$, and to research the iterative solution of the positive eigenvector.

Eq. (1) comes from the research of physics, where the Bose–Einstein condensation of atoms at near absolute zero temperatures is modeled by a nonlinear Schrodinger equation in R^3, also known as the Gross–Pitaevskii equation,

$$-\Delta u + V(x,y,z)u + ku^3 = \lambda u,$$
$$\lim_{|(x,y,z)|\to\infty} u = 0, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,z)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1,$$

where V is a given potential function, (e.g. see [6]). The discretization of this equation leads usually to a nonlinear eigenvalue problem (1). For example, the discretization of the following one dimensional equation:

$$-u''(x) + c(x)u'(x) + ku^3(x) = \lambda u(x), \quad -\infty < x < \infty,$$
$$u(\pm\infty) = 0, \quad \int_{-\infty}^{\infty} u^2(x)\mathrm{d}x = 1.$$

Using finite difference gives a nonlinear eigenvalue problem $Ax + F(x) = \lambda x$ with matrix

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1+\frac{c_1 h}{2} & 0 & \cdots & 0 \\ -1+\frac{c_2 h}{2} & 2 & -1+\frac{c_2 h}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix},$$

and function

$$F(x) = k\begin{bmatrix} x_1^3 \\ x_2^3 \\ \vdots \\ x_n^3 \end{bmatrix},$$

where the c_i's are the values of $c(x)$ at the mesh point x_i, h is the discretization step-size. When h is small enough, A is an M-matrix. This is a nonlinear eigenvalue problem (1).

Remark 1. Eq. (1) becomes $Ax + F(x) = 0$ when $\lambda = 0$. Therefore, $Ax + F(x) = 0$ is a special case of Eq. (1). The theory and solutions of the equation $Ax + F(x) = 0$ have been studied by several researchers, and its basic theoretical properties, such as existence and uniqueness conditions of solution, and some numerical methods of solving this equation have been obtained, e.g., see [4].

2. Preparatory knowledge

The following lemmas about M-matrix can be found in literatures, see, for example [7–11], we present them here to make the paper self-contained.

Definition 1. An M-matrix A is a nonsingsular real matrix,whose off-diagonal entries are nonpositive, and whose inverse has nonnegative entries only.