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On linear combinations of two tripotent, idempotent, and involutive matrices

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Abstract

Let $\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2$, where c_1, c_2 are nonzero complex numbers and $(\mathbf{A}_1, \mathbf{A}_2)$ is a pair of two $n \times n$ nonzero matrices. We consider the problem of characterizing all situations where a linear combination of the form $\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2$ is (i) a tripotent or an involutive matrix when $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting involutive or commuting tripotent matrices, respectively, (ii) an idempotent matrix when $\mathbf{A}_1$ and $\mathbf{A}_2$ are involutive matrices, and (iii) an involutive matrix when $\mathbf{A}_1$ and $\mathbf{A}_2$ are involutive or idempotent matrices.

Keywords: Involutive matrix; Idempotent matrix; Tripotent matrix; Quadratic form; Chi-square distribution; Similar matrix; Diagonalization

1. Introduction and preliminaries

Let $\mathscr{C}$ be the field of complex numbers and $\mathscr{C}^* = \mathscr{C} \setminus \{0\}$. For a positive integer n, let $\mathscr{M}_n$ be the set of all $n \times n$ complex matrices over $\mathscr{C}$. Recall that a matrix $\mathbf{K} \in \mathscr{M}_n$ is idempotent if $\mathbf{K}^2 = \mathbf{K}$, tripotent if $\mathbf{K}^3 = \mathbf{K}$, essentially tripotent if $\mathbf{K}^3 = \mathbf{K}$ with $\mathbf{K}^2 \neq \pm\mathbf{K}$, t-potent if $\mathbf{K}^t = \mathbf{K}$, where t is a positive integer, and involutive if $\mathbf{K}^2 = \mathbf{I}$, where $\mathbf{I}$ stands for the identity matrix. Moreover, $\mathbf{0}$ will mean the zero matrix. Now, consider a linear combination of the form

$$\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2, \tag{1.1}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathscr{M}_n$ are nonzero matrices and $c_1, c_2 \in \mathscr{C}^*$.

For the cases: (i) $\mathbf{A}_1$ and $\mathbf{A}_2$ are idempotent, (ii) $\mathbf{A}_1$ is idempotent and $\mathbf{A}_2$ is tripotent, and (iii) $\mathbf{A}_1$ is idempotent and $\mathbf{A}_2$ is t-potent where $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting matrices, the problem of characterizing some and even all situations where a linear combination of the form (1.1) is an idempotent matrix were studied in [1–4]. It is clear that if $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting matrices, then case (iii) is a natural extension of cases (i) and (ii). Moreover, when $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting tripotent matrices and commuting essentially tripotent matrices, the problem of characterizing some and even all situations where a linear combination of the form (1.1) is a

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tripotent matrix and an idempotent matrix were studied in [5,6]. Furthermore, when $\mathbf{A}_1$ and $\mathbf{A}_2$ are idempotent matrices, the problem of characterizing all situations where a linear combination of the form (1.1) is a group involutory matrix (or, equivalently, a tripotent matrix) were established in [7].

The purpose of this paper is mainly twofold: in case $\mathbf{A}_1$ and $\mathbf{A}_2$ are involutive matrices, to characterize all situations where a linear combination of the form (1.1) is a tripotent or an idempotent or an involutive matrix, and then to determine all situations where a linear combination of the form (1.1) is an involutive matrix when $\mathbf{A}_1$ and $\mathbf{A}_2$ are tripotent or idempotent matrices. The problems should be of interest not only from the algebraic point of view but also from the role these type matrices play in applied sciences, especially statistical theory. For example, the problems considered in this note admit statistical interpretation due to the fact that if $\mathbf{C}$ is an $n \times n$ real symmetric matrix and $\mathbf{x}$ is an $n \times 1$ real random vector having the multivariate normal distribution $N_n(\mathbf{0}, \mathbf{I})$, then necessary and sufficient conditions for the quadratic form $\mathbf{x}'\mathbf{C}\mathbf{x}$ to be distributed as a chi-square variable and as a difference of two independent chi-square variables are that $\mathbf{C} = \mathbf{C}^2$ and $\mathbf{C} = \mathbf{C}^3$, respectively (see e.g. [8–11]). So, $\mathbf{x}$ being as defined above, the problems concerning the tripotency and idempotency of $\mathbf{A}$ of the form (1.1) that considered in this work are related to the questions of when a linear combination of the quadratic forms $\mathbf{x}'\mathbf{A}_1\mathbf{x}$ and $\mathbf{x}'\mathbf{A}_2\mathbf{x}$ is distributed: (i) as a difference of two independent chi-square variables or as a chi-square variable if each of $\mathbf{x}'\mathbf{A}_1\mathbf{x}$ and $\mathbf{x}'\mathbf{A}_2\mathbf{x}$ is distributed as a difference of two independent chi-square variables where $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting real symmetric tripotent matrices, (ii) as a difference of two independent chi-square variables if each of $\mathbf{x}'\mathbf{A}_1\mathbf{x}$ and $\mathbf{x}'\mathbf{A}_2\mathbf{x}$ is distributed as an independent chi-square variable where $\mathbf{A}_1$ and $\mathbf{A}_2$ are commuting real symmetric idempotent matrices.

Now, let us give the following additional concepts and properties. A matrix $\mathbf{B} \in \mathscr{M}_n$ is said to be similar to a matrix $\mathbf{A} \in \mathscr{M}_n$ if there exists a nonsingular matrix $\mathbf{P} \in \mathscr{M}_n$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

If a matrix $\mathbf{A} \in \mathscr{M}_n$ is similar to a diagonal matrix, then $\mathbf{A}$ is said to be diagonalizable. Notice that any tripotent or idempotent or involutive matrix is diagonalizable [12, Corollary 3.3.10]. Applying spectral theorem for diagonalizable matrices in [13], it is obtained that if $\mathbf{A}$ is involutive, then there exist two idempotent matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that $\mathbf{A} = \mathbf{P}_1 - \mathbf{P}_2$, $\mathbf{I} = \mathbf{P}_1 + \mathbf{P}_2$, and $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$. Therefore, the involutiveness leads to the restriction that the sum of the degrees of freedom of different independent quadratic forms must be equal to the dimension of the primary quadratic form matrix in the framework of statistical theory.

The statistical comments given here are subject to the restriction that the matrices are real and symmetric. However, these types of matrices without restrictions are also used in many branches of applied sciences. For example the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which is a member of the class of matrices known as the Pauli spin matrices and the Dirac spin matrices with the submatrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are neither real nor symmetric but involutive, and they are widely used in quantum mechanics (see e.g. [14, pp. 47–51] and [15, p. 495]). In addition, it is noteworthy that there are also important applications of involutive matrices in applied sciences besides the statistical theory (see e.g. [16–18]).

2. Main results

Firstly, we consider the problem of tripotency of linear combination of the form (1.1) where $\mathbf{A}_1$, $\mathbf{A}_2$ are involutive matrices that commute. Meanwhile, due to the fact that an involutive matrix is always a tripotent matrix, it will be useful to start by restating the observation in [5]. Under the assumption that $\mathbf{A}_1$ and $\mathbf{A}_2$ are involutive matrices and $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$, it is clear that $\mathbf{A}_1 = \mathbf{A}_2$ or $\mathbf{A}_1 = -\mathbf{A}_2$. If $\mathbf{A}_1 = \mathbf{A}_2$, then $\mathbf{A}$ is tripotent if and only if

$$c_1 = -c_2 \quad \text{or} \quad c_1 = -c_2 + 1 \quad \text{or} \quad c_1 = -c_2 - 1, \tag{2.1}$$

which corresponds to $\mathbf{A} = \mathbf{0}$, $\mathbf{A} = \mathbf{A}_1$, and $\mathbf{A} = -\mathbf{A}_1$, respectively. If $\mathbf{A}_1 = -\mathbf{A}_2$, then $\mathbf{A}$ is tripotent if and only if

$$c_1 = c_2 \quad \text{or} \quad c_1 = c_2 + 1 \quad \text{or} \quad c_1 = c_2 - 1, \tag{2.2}$$

which also corresponds to $\mathbf{A} = \mathbf{0}$, $\mathbf{A} = \mathbf{A}_1$, and $\mathbf{A} = -\mathbf{A}_1$, respectively.