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Analytic studies and numerical simulations of the generalized Boussinesq equation

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Abstract

The modified Adomian decomposition method is used to solve the generalized Boussinesq equation. The equation commonly describes the propagation of small amplitude long waves in several physical contents. The analytic solution of the equation is obtained in the form of a convergent series with easily computable components. For comparison purposes, a numerical algorithm, based on Chebyshev polynomials, is developed and simulated. Numerical results show that the modified Adomian decomposition method proves to be more accurate and computationally more efficient than the Galerkin–Chebyshev method.

Keywords: Singularly perturbed Boussinesq equation; Decomposition method; Solitary wave solutions; Chebyshev polynomials; Numerical simulations; Boundary-value problems

1. Introduction

The importance of soliton producing nonlinear wave equations is well understood among theoretical physicists and applied mathematicians. An equation admitting soliton solutions which has received comparatively little attention in the literature is

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}. \tag{1.1}$$

It is referred to as the "bad" Boussinesq equation, or the nonlinear beam equation. It describes the motion of long waves in shallow water under gravity in a one-dimensional nonlinear lattices. Eq. (1.1) admits the solitary wave solution

$$u(x,t) = A\ \mathrm{sech}^2(\sqrt{A/6}(x-ct)), \tag{1.2}$$

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where A and $c = \pm\sqrt{1+2A/3}$ are the amplitude and the speed of the solitary wave, respectively. These features of Eq. (1.1) are quite reminiscent of the properties of the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.3}$$

in that they both possess solitary wave solutions, except that the KdV equation allows only one-directional wave propagation and the Boussinesq equation describes bi-directional wave propagation.

In recent years, a great deal of research has been conducted in the study of Eq. (1.1) from various points of view (see, for example, [6,8,10,12] and the references therein). For example, in [10] an exact formula is given for the interaction of solitary waves and in [8], Hirota has deduced conservation laws and has examined N-soliton interaction. The representation of periodic waves as sums of solitons has been given by Whitham in [12]. Wazwaz in [11] used a modified algorithm of the Adomian decomposition method (shortly, ADM) to construct soliton solutions of Eq. (1.1) subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{1.4}$$

El-Sayed and Kaya [6] studied the solitary-wave solutions, using the ADM, of the $(2+1)$-dimensional Boussinesq equation

$$u_{tt} = u_{xx} + u_{yy} + (u^2)_{xx} + u_{xxxx}.$$

In this paper, we consider a regularized version of Eq. (1.1) via the singularly perturbed (sixth-order) Boussinesq equation

$$u_{tt} = u_{xx} + (p(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx}, \tag{1.5}$$

where α and β are real numbers (β is small). This equation was originally introduced by Daripa and Hua [4]. The sixth order derivative term provides dispersive regularization. The physical relevance of Eq. (1.5) in the context of water waves was recently addressed by Dash and Daripa [5]. It was shown that Eq. (1.5) actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water. So, it is closely related to the singularly perturbed (fifth-order) KdV equation

$$u_t + uu_x + u_{xxx} + \epsilon^2 u_{xxxx},$$

which can be derived from Eq. (1.5) by using suitable transformations [5]. The fifth-order KdV equation has been studied by Kaya [9] where soliton solutions were found using the ADM.

Since Eq. (1.1) has solitary wave solutions, the natural question arises whether Eq. (1.5) also admits solitary wave solutions for small values of β. As Eq. (1.5) can serve as a better model than the classical fourth-order Boussinesq equation (1.1)to describe bi-directional wave propagation on the surface of shallow water, in this paper we consider the generalized Boussinesq equation

$$u_{tt} = \sum_{i=0}^{m} b_i u_{(2i+2)x} + [Q(u)]_{xx}, \tag{1.6}$$

where $Q(u) = u + b_0 u^r, r$ and b_i $(i = 1, 2, \ldots, m)$ are all real constants and $u_{(2i+2)x}$ denotes the $(2i+2)$nd derivative of u with respect to x. The modified ADM will be applied to seek soliton solutions of (1.6). Note that the choices $m = 1, b_0 = 1, b_1 = 1$ and $r = 2$ yield Eq. (1.1) and for the choices $m = 2, b_0 = 1, b_1 = \alpha, b_2 = \beta, m = 2$, and $p(u) = u^r$, Eq. (1.6) becomes the singularly perturbed sixth-order Boussinesq Eq. (1.5).

The motivation of this paper is to approach the singularly perturbed Boussinesq equation (1.5) by using the modified ADM [11], the solutions will be calculated in the form of a convergent infinite series with easily computable components. Numerical results will illustrate the rapid convergence of the infinite series.

In order to compare the modified ADM with other existing methods, a Galerkin method based on Chebyshev polynomials is proposed to numerically solve (1.6). The proposed Chebyshev–Galerkin method expresses the solution as a linear combination of Chebyshev polynomials with time dependent coefficients. Using the orthogonality of the Chebyshev polynomials, the partial differential equation is reduced to a coupled system of nonlinear second order ordinary differential equations for the time-dependent expansion coefficients. The second order system of ODEs is further written as a larger system of first-order ODEs which is solved numerically using Range–Kutta method of order 4.