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Numerical integration of some functions over an arbitrary linear tetrahedra in Euclidean three-dimensional space

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Abstract

In this paper it is proposed to compute the volume integral of certain functions whose antiderivates with respect to one of the variates (say either x or y or z) is available. Then by use of the well known Gauss Divergence theorem, it can be shown that the volume integral of such a function is expressible as sum of four integrals over the unit triangle. The present method can also evaluate the triple integrals of trivariate polynomials over an arbitrary tetrahedron as a special case. It is also demonstrated that certain integrals which are nonpolynomial functions of trivariates x, y, z can be computed by the proposed method. We have applied Gauss Legendre Quadrature rules which were recently derived by Rathod et al. [H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, N.L. Ramesh, Gauss Legendre Quadrature over a Triangle, J. Indian Inst. Sci. 84 (2004) 183–188] to evaluate the typical integrals governed by the proposed method.

Keywords: Finite element method; Numerical integration; Gauss Legendre quadrature; Tetrahedron; Triangular elements; Extended numerical integration

1. Introduction

Volume, center of mass, moment of inertia and other geometric properties of rigid homogeneous solids frequently arise in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modelling as well as in robotics. Integration formulas for multiple integrals have always been of great interest in computer applications, a good overview of available methods for evaluating volume integrals is given by Lee and Requicha [2]. Timmer and Stern [3] discussed a theoretical approach to the evaluation of volume integrals by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajiya [4] presented an outline of a closed form formula for volume integration by decomposing the solid into a set of solid tetrahedra. Cattani and paoluzzi [5] gave a symbolic solution to both the surface and volume integration of polynomials by using a triangulation of the solid boundary. In a recent paper, Bernardini [6] has presented explicit formulas and algorithms for computing integrals of polynomials over

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n-dimensional polyhedra by using the decomposition representation and the boundary representation of the polyhedron. Rathod and Govind Rao [7] derived explicit integration formulas for computing volume integrals of trivariate polynomials over an arbitrary tetrahedron in Euclidean three-dimensional space. They proposed two different approaches; the first method evaluates this volume integral by mapping the tetrahedron into orthogonal unit tetrahedron and the second method computes the same integral as a sum of four integrals over the unit triangle. The present work enhances the second method by considering the evaluation of some functions by use of the well known Gauss Divergence theorem.

2. Problem statement for present work

Most computational studies of triple integrals deal with problems in which the domain of integration is very simple, like a cube or a sphere, but the integrand is complicated. However, in real applications, we confront the inverse problem: the integrating function $f(x,y,z)$ is usually simple; but the domain is very complicated. Hence in this paper and in other previous works [4,6] an attempt is made to obtain practical formulas for the exact evaluation of integrals.

$$\int\int_P\int f(x,y,z)\mathrm{d}V,$$

where P is a three-polyhedron in R^3 and $\mathrm{d}V$ is the differential volume. The integrating-function is a trivariate monomial

$$f(x,y,z)=x^{\alpha}y^{\beta}z^{\gamma},$$

where α,β,γ are nonnegative integers

or

$$f(x,y,z)=\frac{\partial F}{\partial x},\quad \text{or}\quad f(x,y,z)=\frac{\partial F}{\partial y},\quad \text{or}\quad f(x,y,z)=\frac{\partial F}{\partial z}.$$

However the paper is focused on the calculation of the following integral:

$$III_V=\int\int_V\int f(x,y,z)\mathrm{d}V,$$

where $f(x,y,z)=\frac{\partial F}{\partial x}$, or $f(x,y,z)=\frac{\partial F}{\partial y}$, or $f(x,y,z)=\frac{\partial F}{\partial z}$, for some suitable function F and V is an arbitrary tetrahedron with four vertices (x_i,y_i,z_i), $(i=1,2,3,4)$. Two different approaches are possible. The first approach is direct and it transforms an arbitrary tetrahedron into an orthogonal tetrahedron by means of a mapping. The second approach is based on the fact that certain triple integrals can be reduced to surface integrals by use of the well known Gauss's divergence theorem. This paper is concerned with the second approach.

3. Volume integration over an arbitrary tetrahedron

In this section, we first obtain the volume integral of a scalar function

$$f(p)=f(x,y,z)$$

(α,β,γ are positive integers) over an arbitrary tetrahedron by transforming it to an orthogonal unit tetrahedron. That is we are actually interested in evaluating

$$III_V=\int\int_V\int f(x,y,z)\mathrm{d}V, \tag{1}$$

where V is an arbitrary tetrahedron in the x,y,z Cartesian coordinate system.