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Some implicit methods for the numerical solution of Burgers' equation

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Abstract

In this article we develop multisymplectic box methods on Burgers' equation. Burgers' equation has some discontinuous solutions because of effects of viscosity term. These discontinuities raise phenomenon of shock waves. Some explicit and implicit numerical methods have been experimented on Burgers' equation but these schemes have not been seen proper in this case because of their conditional stability and existence of viscosity term. To avoid these problems, multisymplectic box scheme which has been constructed for KdV equation is purposed for Burgers' equation. Multisymplectic box scheme is a very effective box scheme in diminishing artificial wiggles which appear in approximation solutions. We consider two types of box schemes and implement on Burgers' equation to get better results with no artificial wiggles.

Keywords: Box scheme; Multisymplectic method; Burgers' equation; Implicit scheme; Viscous term

1. Introduction

The study of partial differential equations (PDEs) started in the 18th century in the work of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science [3]. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDEs also became an essential tool in other branches of mathematics [3]. One of the most important PDEs in the theory of nonlinear conservation laws, is Burgers' equation. This equation has a large variety of applications in modeling of water in unsaturated soil, dynamic of soil water, statistics of flow problems, mixing and turbulent diffusion, cosmology and seismology [9–12]. The Burgers' equation is a nonlinear equation, very similar to the Navier–Stokes equation which could serve as a nonlinear analog of the Navier–Stokes equations. This single equation has a convection term, a diffusion term and a time-dependent term:

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$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}.$$

Burgers' equation is parabolic when the viscous term is included. If the viscous term is neglected, the remaining equation is hyperbolic. If the viscous term is dropped from the Burgers' equation the nonlinearity allows discontinuous solutions to develop. In Burgers' equation discontinuities may appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves with important applications in physics [3]. These properties make Burgers' equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated.

Discretization methods are well-known techniques for solving Burgers' equation. One of the most simple one is leap-frog explicit scheme [10] which was proposed in the 1960s. This explicit scheme is very easy to formulate but fails to give a correct solution when the viscosity is too small. To avoid these unstable conditions, implicit methods such as Cranck–Nicolson type scheme is presented, but this scheme cannot be used for very small viscosities [12].

On the other hand, it has long been known that conservative discretization schemes for nonlinear, nondissipative partial differential equations (PDEs) governing wave phenomena are numerically unstable, and dissipation has subsequently been routinely introduced into such numerical schemes [1]. For nonlinear problems of this type, in particular, conservative difference schemes are known to occasionally yield numerical solutions which at first look fine, but at a later time may suddenly explode [1].

Consequently, nondissipative schemes are impractical, especially for long time integration. In the context of Hamiltonian systems this corresponds to using a slightly dissipative discretization scheme.

Ascher and McLachan established a method based on box scheme [2]. The name of "box scheme" is a general name for several numerical schemes of different origin [7]. The box scheme, a compact scheme in both x and t centered at cell (box), has been used for many years [1]. For example Keller [8] introduced this scheme on the 1-D heat equation in the 1970s. Box schemes types are conservative schemes, i.e. schemes which guarantee, for equations in divergence form, an exact conservation of the flux at the level of the box. The first one has been introduced in the 1980' in compressible computational fluid dynamics. As in Keller's scheme, the basic idea is that locating the degrees of freedom at the center of the faces instead at the center of the cells, could be more interesting for an accurate evaluation of conservative fluxes, [4–6]. In this paper whose inspiration is [1], we consider some multisymplectic box methods for the Burgers' equation. To this aim we examine this situation numerically, attempting to see whether carefully designed, conservative finite difference and finite volume discretizations can remain stable and deliver sharp solution profiles in comparison to other methods which examined before. Then the obtained results are comparable with results of explicit and Crank–Nicolson. Indeed, there is an entire family of box schemes with similar stability and accuracy properties to those of the multisymplectic box scheme.

2. Numerical methods

2.1. Some discretization methods

The Burgers' equation is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

We consider this equation in the case of initial condition with periodic boundary condition. There are several numerical methods for solving Burgers' equation based on discritization on a fixed grid for both space and time variables. The following discritization scheme is explicit leap-frog scheme [10] which was proposed in 1960:

$$u_i^{n+1} = u_i^{n-1} + \frac{\mu}{6}(u_{i-1}^n + u_i^n + u_{i+1}^n)(u_{i+1}^n - u_{i-1}^n) + \frac{v\mu}{\Delta x}(u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{2}$$

In the above formula the artificial boundary at $n = 0$ is approximated by $u_i^{-1} = u_i^1$. By approximating the artificial boundary with extrapolation methods along characteristics, better results can be extracted [13].