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A GMRES-based BDF method for solving differential Riccati equations

Vicente Hernández [a], Jacinto Javier Ibáñez [a], Jesús Peinado [a,*], Enrique Arias [b]

[a] *Departamento de Sistemas Informáticos y Computación, Technical University of Valencia, Camino de Vera s/n, 46022 Valencia, Spain*
[b] *Departamento de Sistemas Informáticos, University of Castilla-La Mancha, Avda. España s/n, 02071 Albacete, Spain*

Abstract

Differential Riccati equations play a fundamental role in control theory, for example, optimal control, filtering and estimation, decoupling and order reduction, etc. The most popular codes to solve stiff differential Riccati equations use backward differentiation formula (BDF) methods. In this paper, a new approach to solve differential Riccati equations by means of a BDF method is described. In each step of these methods an algebraic Riccati equation is obtained, which is solved by means of Newton's method. In the standard approach, this system is transformed into a Sylvester equation, which could be solved by means of the well-known Bartels–Stewart method. In our code, we obtain a system of linear equations, defined from a Kronecker product of matrices related to coefficient matrices of the differential Riccati equation, that is solved by means of the iterative generalized minimum residual (GMRES) method. We have also implemented an efficient matrix–vector product in order to reduce the computational and storage cost of the GMRES method. The above approach has been applied in the development of an algorithm to solve differential Riccati equations. The accuracy and efficiency of this algorithm has been compared with the BDF algorithm that uses the Bartels–Stewart method. Experimental results show the advantages of the new algorithm.

Keywords: Differential Riccati equations; BDF methods; GMRES methods; Algebraic Riccati Equation

1. Introduction

Consider the differential Riccati equation (DRE)

$$\begin{aligned} \dot{X}(t) &= A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t), \\ X(t_0) &= X_0, \quad t_0 \leqslant t \leqslant t_f, \end{aligned} \tag{1}$$

where $A_{11}(t) \in R^{n\times n}$, $A_{22}(t) \in R^{m\times m}$, $A_{12}(t) \in R^{n\times m}$, $A_{21}(t) \in R^{m\times n}$ and $X(t) \in R^{m\times n}$.

* Corresponding author.
E-mail addresses: vhernand@dsic.upv.es (V. Hernández), jjibanez@dsic.upv.es (J.J. Ibáñez), jpeinado@dsic.upv.es (J. Peinado).

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The DRE arises in several applications, in particular in control theory, for example the time-invariant linear quadratic optimal control problem. In this case, the DRE has the following expression:

$$\dot{X}(t) = Q + A^{T}X + XA - XBR^{-1}B^{T}X, \tag{2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $Q = Q^{T} \in R^{n\times n}$ is positive semidefinite, and $R = R^{T} \in R^{m\times m}$ is positive definite, representing, respectively, the state matrix, the input matrix, the state weight matrix and the input weight matrix. Another application of the DRE (1) consists of solving a two point boundary problem, by decoupling this problem in two initial value problems [1].

This work is concerned with the study and implementation of a method for solving the DRE (1) by numerical integration, using the BDF method. The result is an implicit scheme which solves the discrete version of the DRE [1] according to the property that the discretization of a polynomial differential equation reduces it to a polynomial algebraic equation of the same degree.

An example of this methodology appeared in [1], known as DRESOL. This package is based on LSODE software, developed by Hindsmarsh [2]. Several methods have been implemented for solving the algebraic Riccati equation (ARE); however, in the context of stiff DREs, one of the better choices for solving the associated ARE is to apply implicit schemes based on Newton's or quasi-Newton's methods. In both cases, at each iteration step a Sylvester equation has to be solved

$$G_{11}Y - YG_{22} = H, \tag{3}$$

where $G_{11} \in R^{n\times n}$, $G_{22} \in R^{m\times m}$ and $H \in R^{n\times m}$ change at each iteration if Newton's method is used.

A standard method for solving Eq. (3) is Bartels–Stewart algorithm [3]. By using this approach, matrices G_{11} and G_{22} are both reduced via orthogonal matrices U and V to real Schur form, obtaining the equivalent equation

$$(U^{T}G_{11}U)(U^{T}YV) - (U^{T}YV)(V^{T}G_{22}V) = U^{T}HV. \tag{4}$$

Once the quasi-triangular problem (4) is solved for $U^{T}YV$, then Y is easily recovered.

This paper is organized as follows. First, Section 2 describes the numerical integration method using BDF and a methodology as explained in [1] as a starting point. Section 3 presents our approach for solving DRE applying BDF and GMRES methods. A theoretical study in terms of memory storage and flops requirements is included. A sequential implementation of the method has been carried out using standard linear algebra libraries such as basic linear algebra subroutines (BLAS) [4] and linear algebra package (LAPACK) [5]. Section 4 presents the test battery and the experimental results. Finally, the conclusions and future work are outlined in Section 5.

2. Numerical integration using BDF

The DRE occurs in several applications in different fields of science and engineering. It did not however receive enough attention in the numerical literature until the mid seventies. Since then many different methods have been proposed [6,7]. These methods can be grouped into several classes: Vectorized approach, linearization approach [8–10], Chandrasekhar approach [11], superposition methods [12,13], BDF methods [14,15,1,16], and Hamiltonian approach [17,18].

Let the Riccati equation

$$\dot{X}(t) = F(t,X), \quad X(t_0) = X_0, \quad t \in [t_0, t_f], \tag{5}$$

where

$$F(t,X) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t),$$

with $A_{11}(t) \in R^{n\times n}$, $A_{12}(t) \in R^{n\times m}$, $A_{21}(t) \in R^{m\times n}$, $A_{22}(t) \in R^{m\times m}$ and $X(t) \in R^{m\times n}$.

In the literature it is possible to find several methods for solving this equation. One of these methods is the well-known backward differentiation formula (BDF) [19]. With a BDF scheme, the integration