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A secret to create a complete market from an incomplete market

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Abstract

The Martingale method has been given increasing attention since it was conducted by Cox and Huang in 1989. Martingale method allows us to solve the problems of utility maximization in a very elegant manner. However, the Martingale method is not omnipotent. When the market is incomplete, traditional Martingale method will be problematic. To overcome the problem of incompleteness, Karatzas/Shreve/Xu [Martingale and duality for utility maximization in an incomplete market, SIAM J. Control and Optimization 29 (1991) 702–730] have developed a way to complete the market by introducing additional *fictitious* stocks and then making them uninteresting to the investor. Nevertheless, to find such fictitious stocks is not straightforward. In particular, when the number of such stocks needed in order to complete the market were very large, it would be very computational, and even may not be possible to be expressed explicitly. To make life easier, we provide an alternative method by directly creating a complete market from the incomplete one such that the dimension of the underlying Brownian motion equals the number of available stocks. Our approach is ready to be used.

Keywords: Incomplete market; Martingale method; Optimal portfolio; Continuous time

1. Introduction

We assume that, in a continuous-time financial market, an investor with an initial capital x invests his wealth into a riskless bond and m stocks whose prices are driven by a d-dimensional Brownian motion. He wishes to maximize his *expected utility of final wealth* for a given utility function but has no knowledge about future prices and has no inside information either. His optimal decision is made only by observing the past and the present stock prices. In a complete market, his optimal trading strategy can be solved by the traditional Martingale method based on the fact that every desired final wealth can be obtained by trading following an appropriate portfolio strategy given enough initial wealth in a complete financial market. Incompleteness arises when the number of stocks is strictly less than the dimension of the underlying Brownian motion. In such a situation, the traditional Martingale method cannot solve the investor's maximization problem directly. To overcome the problem of incompleteness, Karatzas et al. [1] have developed a way to complete the market by introducing additional *fictitious* stocks and then making them uninteresting to the investor so that the

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optimal proportions of his wealth invested into such stocks are actually equal nothing. In this paper, we introduce an easier way to transform an incomplete market to a complete one so that the traditional Martingale method works. Our principal is reducing the dimension of the driving Brownian motion by summing up the normalized Itô integrals to get new Brownian motions. However, the created new Brownian motions are no longer independent. We need to recreate independent Brownian motions from correlated ones. These procedures are demonstrated in Section 4, in which the optimal solution to the investor's maximization problem in the incomplete market is presented.

This paper is structured as following. Section 2 describes the general framework of a continuous-time financial market consisting of one riskless bond and m stocks whose prices are driven by a d-dimensional Brownian motion. Section 3 presents the optimal portfolio problem and its solution solved by the Martingale method. Our key result is presented in Section 4. Section 5 concludes the paper.

2. The continuous-time financial model

We consider a financial market consisting of one riskless bond with price $S_0(t)$ given by

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\,\mathrm{d}t, \quad S_0(0) = 1 \tag{2.1}$$

and m stocks with prices $S_i(t)$ satisfying

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = b_i(t)\,\mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}(t)\,\mathrm{d}W_j(t), \quad i = 1, \ldots, m, \tag{2.2}$$

where $W(t) = (W_1(t), \ldots, W_d(t))^{\mathrm{T}}$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the component Brownian motions $W_j(t), j = 1, \ldots, d$, being independent. The interest rate $r(t)$, the stock appreciation rate vector $b(t) := (b_1(t), \ldots, b_m(t))^{\mathrm{T}}$ and the volatility matrix $\sigma(t) := \{\sigma_{ij}(t)\}_{m\times d}$ are assumed to be either deterministic or functions of the asset prices $S_i(t)$, for $i = 0, 1, \ldots, m$. In addition, the matrix $\sigma(t)$ is assumed to have full row rank.

Under the assumptions above, we restrict ourselves to the situation where $m \leqslant d$, i.e., the number of stocks in the market is smaller or equal to the dimension of the underlying Brownian motion. This financial market is *complete* when $m = d$ while *incompleteness* arises if $m < d$.

We assume that an investor has initial wealth $x > 0$ and invests a proportion of his wealth $\pi_i(t)$ into the ith stock at time t, $t \in [0, T]$, for $i = 1, \ldots, m$. Furthermore, we assume that the investor has no exterior funding. Denote by $\pi(t) := (\pi_1(t), \ldots, \pi_m(t))^{\mathrm{T}}$, the *portfolio process* is hence *self-financing*. The corresponding *wealth process* $X^{\pi}(t)$ is derived as follows:

$$\begin{aligned}\frac{\mathrm{d}X^{\pi}(t)}{X^{\pi}(t)} &= (1 - \pi^{\mathrm{T}}(t)\mathbf{1}_m)r(t)\,\mathrm{d}t + \pi^{\mathrm{T}}(t)[b(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)] \\ &= r(t)\,\mathrm{d}t + \pi^{\mathrm{T}}(t)[(b(t) - r(t)\mathbf{1}_m)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)],\end{aligned} \tag{2.3}$$

where $\mathbf{1}_m := (1, \ldots, 1)^{\mathrm{T}}$ is an $m \times 1$ vector. For simplicity, we simply write $X(t)$ for $X^{\pi}(t)$ in what follows, unless otherwise stated. For the case when $m = d$, we define the *market price of risk* $\theta(t)$ by

$$\theta(t) := \sigma^{-1}(t)(b(t) - r(t)\mathbf{1}_m). \tag{2.4}$$

Then, the wealth process can be rewritten as

$$\frac{\mathrm{d}X(t)}{X(t)} = r(t)\,\mathrm{d}t + \pi^{\mathrm{T}}(t)\sigma(t)[\theta(t)\,\mathrm{d}t + \mathrm{d}W(t)]. \tag{2.5}$$

Proposition 2.1. *Let us define the discount factor $H(t)$ by*

$$H(t) := \exp\left\{-\int_0^t r(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,\mathrm{d}s - \int_0^t \theta^{\mathrm{T}}(s)\,\mathrm{d}W(s)\right\} \tag{2.6}$$

then $H(\cdot)X(\cdot)$ is a Martingale under the actual probability measure $\mathbb{P}$.