Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics and Computation 193 (2007) 73–88

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Existence and iterative approximations of nonoscillatory solutions of higher order nonlinear neutral delay differential equations

Zeqing Liu [a,*], Shin Min Kang [b], Jeong Sheok Ume [c]

[a] *Department of Mathematics, Liaoning Normal University, P.O. Box 200, Dalian, Liaoning 116029, People's Republic of China*
[b] *Department of Mathematics and The Research Institute of Natural Science, Gyeongsang National University, Chinju 660-701, South Korea*
[c] *Department of Applied Mathematics, Changwon National University, Changwon 641-733, South Korea*

Abstract

This paper studies a few existence results of nonoscillatory solutions for the following higher order nonlinear neutral delay differential equation:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+cx(t-\tau)]+(-1)^{n-i+1}\frac{\mathrm{d}^i}{\mathrm{d}t^i}h(t,x(h_1(t)),x(h_2(t)),\ldots,x(h_k(t)))$$
$$+(-1)^{n+1}f(t,x(f_1(t)),x(f_2(t)),\ldots,x(f_k(t)))=g(t),\quad t\geqslant t_0,$$

where n is a positive integer, $0\leqslant i\leqslant n-1$, $c\in\mathbb{R}\setminus\{-1\}$, $\tau>0$, $h\in C^i([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$, $f\in C([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$, $g\in C([t_0,+\infty),\mathbb{R})$, $h_l\in C^i([t_0,+\infty),\mathbb{R})$ and $f_l\in C([t_0,+\infty),\mathbb{R})$ with

$$\lim_{t\to+\infty}h_l(t)=\lim_{t\to+\infty}f_l(t)=+\infty,\quad l=1,\ldots,k$$

constructs several Mann type iterative approximation sequences with errors for these nonoscillatory solutions and establish some error estimates between the approximate solutions and the nonoscillatory solutions. In addition sufficient conditions for the existence of infinitely many nonoscillatory solutions for the above equation are given. Three nontrivial examples are given to illustrate the advantages of our results.

Keywords: Higher order nonlinear neutral delay differential equation; Nonoscillatory solution; Infinitely many nonoscillatory solutions; Contraction mapping; Mann iterative sequence with errors

1. Introduction and preliminaries

Many researchers [1–11,13–22] and others studied the existence and asymptotic behaviors of nonoscillatory and oscillatory solutions for the following several classes of neutral delay differential equations:

* Corresponding author.
E-mail addresses: zeqingliu@dl.cn (Z. Liu), smkang@nongae.gsnu.ac.kr (S.M. Kang), jsume@sarim.changwon.ac.kr (J.S. Ume).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.03.040

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-cx(t-\tau)]+a\int_0^{+\infty}k(s)x(t-s)\mathrm{d}s=0,\quad t\geqslant t_0;\tag{1.1}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+cx(t-\tau)]+ax(t-\sigma)=0,\quad t\geqslant t_0;\tag{1.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+cx(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geqslant t_0;\tag{1.3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[x(t)+cx(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geqslant t_0;\tag{1.4}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)]=0,\quad t\geqslant t_0.\tag{1.5}$$

Sficas and Stavroiclakis [14] and Gopalsamy and Lalli [6] gave some sufficient and necessary conditions, which guarantee all solutions of Eq. (1.1) to have at least one zero. Ladas and Sficas [11], Qian [13] and Bainov and Petrov [3] investigated, respectively, the asymptotic behavior of nonoscillatory solutions for Eq. (1.2). In 1998 and 2001, Kulenović and Hadžiomerspahić [9,10] established the existence of nonoscillatory solutions of Eqs. (1.3) and (1.4) under the assumptions $c\neq\pm1$, $aQ_1(t)\geqslant Q_2(t)$ and other conditions. In 2002 and 2003, Zhou and Zhang [22] extended the results in [9,10] from Eqs. (1.3) and (1.4) to Eq. (1.5). In 2004, Cheng and Annie [4] continued to study the existence of nonoscillatory solutions for Eq. (1.4) by omitting the conditions $c\neq1$ and $aQ_1(t)\geqslant Q_2(t)$, which were used by Kulenović and Hadžiomerspahić [10]. However, all papers mentioned above only dealt with the existence and asymptotic behaviors of nonoscillatory solutions of Eqs. (1.1)–(1.5), and did not suggest the iterative approximations of these nonoscillatory solutions and the existence of infinitely many nonoscillatory solutions for of (1.1)–(1.5).

Our aim in this paper is to investigate the following higher order nonlinear neutral delay differential equation:

$$\begin{aligned}&\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+cx(t-\tau)]+(-1)^{n-i+1}\frac{\mathrm{d}^i}{\mathrm{d}t^i}h(t,x(h_1(t)),x(h_2(t)),\ldots,x(h_k(t)))\\&\quad+(-1)^{n+1}f(t,x(f_1(t)),x(f_2(t)),\ldots,x(f_k(t)))=g(t),\quad t\geqslant t_0,\end{aligned}\tag{1.6}$$

where n is a positive integer, $0\leqslant i\leqslant n-1$, $c\in\mathbb{R}\setminus\{-1\}$, $\tau>0$, $h\in C^i([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$, $f\in C([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$, $g\in C([t_0,+\infty),\mathbb{R})$, $h_l\in C^i([t_0,+\infty),\mathbb{R})$ and $f_l\in C([t_0,+\infty),\mathbb{R})$ with

$$\lim_{t\to+\infty}h_l(t)=\lim_{t\to+\infty}f_l(t)=+\infty,\quad l=1,\ldots,k.$$

By making use of the contraction mapping principle, we give several existence results of nonoscillatory solutions for Eq. (1.6), construct a few Mann type iterative approximation schemes with errors for these nonoscillatory solutions, discuss error estimates between the approximate solutions and the nonoscillatory solutions and establish two existence results of infinitely many nonoscillatory solutions for Eq. (1.6). These results presented in this paper extend, improve and unify many known results due to Cheng and Annie [4], Graef, Yang and Zhang [7], Kulenović and Hadžiomerspahić [9,10], Zhang [17], Zhang and Yu [19], and Zhou and Zhang [22] and others. Three nontrivial examples are given to illustrate the advantages of our results.

By a solution of Eq. (1.6), we mean a function $x\in C([t_1-\tau,+\infty),\mathbb{R})$ for some $t_1\geqslant t_0$, such that $x(t)+cx(t-\tau)$ is n times continuously differentiable on $[t_1,+\infty)$ and such that Eq. (1.6) is satisfied for $t\geqslant t_1$. As is customary, a solution of Eq. (1.6) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise. It is assumed throughout this paper that

(H_1) there exist constants $M>N>0$ and functions $p,q,r,w\in C([t_0,+\infty),\mathbb{R}^+)$ satisfying

$$\begin{aligned}&|f(t,u_1,u_2,\ldots,u_k)-f(t,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)|\leqslant p(t)\ \max\{|u_l-\bar{u}_l|:1\leqslant l\leqslant k\},\\&|h(t,u_1,u_2,\ldots,u_k)-h(t,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)|\leqslant r(t)\ \max\{|u_l-\bar{u}_l|:1\leqslant l\leqslant k\}\end{aligned}$$