Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics and Computation 198 (2008) 24–34

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

The (M, N)-symmetric Procrustes problem

Juan Peng [a,*], Xi-Yan Hu [b], Lei Zhang [b]

[a] College of Applied Sciences, Beijing University of Technology, Beijing 100022, PR China
[b] College of Mathematics and Econometrics, Hunan University, Changsha 410082, PR China

Abstract

An $p \times q$ matrix A is said to be (M,N)-symmetric if $MAN = (MAN)^{\mathrm{T}}$ for given $M \in R^{n\times p}, N \in R^{q\times n}$. In this paper, the following (M,N)-symmetric Procrustes problem is studied. Find the (M,N)-symmetric matrix A which minimizes the Frobenius norm of $AX - B$, where X and B are given rectangular matrices. We use Project Theorem, the singular-value decomposition and the generalized singular-value decomposition of matrices to analysis the problem and to derive a stable method for its solution. The related optimal approximation problem to a given matrix on the solution set is solved. Furthermore, the algorithm to compute the optimal approximate solution and the numerical experiment are given.

Keywords: (M,N)-symmetric matrix; Procrustes problem; Optimal approximation

1. Introduction

Let R^n denote the set of n-dimensional real vectors, and $R^{n\times n}, SR^{n\times n}, ASR^{n\times n}, OR^{n\times n}$ denote the set of real $n \times n$ matrices, real $n \times n$ symmetric matrices, real $n \times n$ antisymmetric matrices and orthogonal $n \times n$ matrices, respectively. The notation $A^+, \|A\|$ stands for the Moore-Penrose inverse and the Frobenius norm of a matrix A, respectively. For $A = (a_{ij}) \in R^{n\times m}$ and $B = (b_{ij}) \in R^{n\times m}$, define $A * B = (a_{ij}b_{ij}) \in R^{n\times m}$ as Hadamard product of matrices A and B.

Definition 1. Given $M \in R^{n\times p}, N \in R^{q\times n}$, we say that $A \in R^{p\times q}$ is (M,N)-symmetric if

$$MAN = (MAN)^{\mathrm{T}}.$$

We denote by $SR^{p\times q}(M,N)$ the set of all (M,N)-symmetric matrix.

In this paper, we consider the following problems:

Problem I. Given $M \in R^{n\times p}, N \in R^{q\times n}, X \in R^{q\times k}, B \in R^{p\times k}$, find $A \in SR^{p\times q}(M,N)$ such that

$$\|AX - B\| = \min.$$

* Corresponding author.
E-mail address: pengjuanhn@yahoo.com.cn (J. Peng).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.08.094

Problem II. Given $A^* \in R^{p\times q}$, find $\widehat{A} \in S_E$ such that

$$\|\widehat{A} - A^*\| = \min_{A\in S_E} \|A - A^*\|,$$

where S_E is the solution set of Problem I.

In electricity, control theory and processing of digital signals, we often need to study the Procrustes problem of the well-known equation $AX = B$ with the unknown matrix A. The problem with A being symmetric, bisymmetric, centrosymmetric and positive semidefinite symmetric were studied (see [1–8]). Taking for A the set of orthogonal matrices yields the orthogonal Procrustes problem (see [9]), the set of symmetric matrices yiedls the symmetric Procrustes problem (see [10]). By analogy with the problem mentioned above, we will refer to Problem I as the (M,N)-symmetric Procrustes problem. Usually, by applying the structure properties of unknown matrix A and appropriate matrix decompositions (the singular-value decomposition, the generalized singular-value decomposition, etc.), the solution of the Procrustes problem were given. But this approach can not be used to solve Problem I. In this paper, we initiate an efficient method: Firstly, by the structure property of A and a set of orthonormal basis of a subspace, we find out a solution A_0 of Problem I. Secondly, using the solution A_0 and the Project Theorem, we transfer Problem I to the problem of finding the (M,N)-symmetric solution of a consistent matrix equation $AX = A_0X$. Finally, we find out the (M,N)-symmetric solution of the consistent matrix equation.

Problem II, that is, the optimal approximation problem of a matrix with the given matrix restriction, is proposed in the processes of test or recovery of linear systems due to incomplete data or revising given data. A preliminary estimate A^* of the unknown matrix A can be obtained by the experimental observation values and the information of statical distribution. The optimal estimate of A is a matrix $\widehat{A}$ satisfying the given matrix restriction for A and being the best approximation of A^*. Various aspects of the optimal approximation problem associated with $AX = B$ were considered, see [1,2] and reference therein.

The paper is organized as follows. In Section 2, we will discuss the structure properties of matrices in $SR^{p\times q}(M,N)$, and provide the general expression of the solutions of Problem I. In Section 3, we will prove the existence and uniqueness of the solution of Problem II and derive the expression of this unique solution. In Section 4, we will give the algorithm to compute the approximate solution and the numerical experiment.

2. The general form of solutions of Problem I

For convenience, here we give the singular-value decomposition (SVD) of a matrix X, and the generalized singular-value decomposition (GSVD) of a matrix pair $[M^T, N]$. Proofs and properties concerning the SVD and GSVD can be found in [11–13].

Given a matrix $X \in R^{q\times k}$ of rank r, its SVD is

$$X = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T = U_1 \Sigma V_1^T, \tag{1}$$

where $\begin{pmatrix} \underset{r}{U_1}, & \underset{q-r}{U_2} \end{pmatrix} \in OR^{q\times q}$, $\begin{pmatrix} \underset{r}{V_1}, & \underset{k-r}{V_2} \end{pmatrix} \in OR^{k\times k}$, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) > 0$.

Given two matrices $M \in R^{n\times p}, N \in R^{q\times n}$, the GSVD of the matrix pair $[M^T, N]$ is

$$M^T = \widetilde{U}\Sigma_1 W^T, \qquad N = \widetilde{V}\Sigma_2 W^T, \tag{2}$$

where W is a nonsingular $n \times n$ matrix, $\widetilde{U} \in OR^{p\times p}$, $\widetilde{V} \in OR^{q\times q}$, and

$$\Sigma_1 = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & \Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} t \\ s \\ p-t-s \end{matrix}, \qquad \Sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \end{pmatrix} \begin{matrix} q-l+t \\ s \\ l-t-s \end{matrix}. \tag{3}$$

$$\begin{matrix} t & s & l-t-s & n-l \end{matrix} \qquad\qquad \begin{matrix} t & s & l-t-s & n-l \end{matrix}$$

Here, $l = \mathrm{rank}((M, N^T))$, $t = l - \mathrm{rank}(N)$, $s = \mathrm{rank}(M^T) + \mathrm{rank}(N) - l$, and $\Omega_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_s)$, $\Omega_2 = \mathrm{diag}(\beta_1, \ldots, \beta_s)$ with $1 > \alpha_1 \geqslant \cdots \geqslant \alpha_s > 0$, $0 < \beta_1 \leqslant \cdots \leqslant \beta_s < 1$, and $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, 2, \ldots, s$.