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The characterizations of optimal solution set in programming problem under inclusion constrains

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Abstract

The characterizations of the solution set in extremal problem under inclusion constrains:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in C, \quad 0 \in F(x) \end{array} \tag{P}$$

is considered in this paper. When f is continuously convex and F is a set-valued map with convex graph, the Lagrange function of (P) is proved to be a constant on the solution set, and this property is then used to derive various simple Lagrange mulitiplier-based characterizations of the solution set of (P).

Keywords: Inclusion constrains; Support function; Subgradient; Solution set

1. Introduction

The characterization of optimal solution of a mathematical programming is an important study in optimization problems, and it is fundamental for the development of solution methods. It can be widely used in applied mathematics fields [1,2].

Jeyakumar [3] presented characterization of the solution sets of the following cone-constrained convex programming

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in C, \quad -g(x) \in K, \end{array} \tag{P$'$}$$

where X and Y are Bnanach Spaces, C is a closed convex subset of X, K is a closed convex cone in Y, $f : X \to R$ is a continuous convex function, and $g : X \to Y$ is a continuous K mapping. The Lagrange multiplier, which is key to identifying optimal solution for constrained optimization, is used to characterize the solution set of (P′).

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First, the author established that the Lagrange function of (P′) is constant on the solution set of (P′). Then, he used this elementary property to present various simple Lagrange multiplier-based characterizations of the solutions set of (P′).

In this paper, we consider the programming problem under inclusion constrains:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in C, \quad 0 \in F(x). \end{aligned} \tag{P}$$

Suppose that C is a closed convex subset of X, f is a continuous convex function and g is a set-valued mapping with convex graph. Obviously, the constrain $-g(x) \in K$ can be written $0 \in g(x) + K$. It is also easy to derive that $g(x) + K$ is a map with convex graph on C, that is to say, the problem (P′) is a special case of problem (P) where $F(x) = g(x) + K$ (Remark 2.1). We prove that the Lagrange function of problem (P) is constant on is solution set (Theorem 3.2). And we derive various characterizations of the solution set using the Lagrange multiplier (Theorem 3.3, Propositions 3.1, 3.2 and 3.3).

2. Preliminaries

Let X and Y to be Banach Spaces, and X^* and Y^* are their dual spaces. Let C to be a nonempty closed subset of X. Suppose that $f : X \to R$ is a real-valued function and that $g : X \to Y$ is a set-valued mapping.

Definition 2.1. A function f is said to be satisfy a *Lipschitz condition* of rank K on a given set C provided that F is finite on C and satisfies

$$|f(x) - f(y)| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

A function f is said to be *Lipschitz near* x if it satisfis the Lipschitz condition on a neighborhood of x. A function f is said to be *Locally Lipschitz on* C if f is Lipschitz near x for every $x \in C$.

Definition 2.2. Let f be Lipschitz of rank K near a given point $x \in X$. The *generalized directional derivative* of f at x in the direction v, denoted $f^\circ(x;v)$, is defined as follows:

$$f^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

where of course y is a vector in X and t is a positive scalar.

Definition 2.3. The *generalized gradient* of f at x, denoted $\partial f(x)$, is defined to be the subset of X^*

$$\partial f(x) = \{x^* \in X^* : f^\circ(x; v) \geqslant \langle x^*, v\rangle, \quad \forall v \in X\}.$$

Proposition 2.1. *Let f be convex on C and Lipschitz near $x \in C$. Then the directional derivatives $f'(x; v)$ exist, and we have $f'(x;v) = f^\circ(x;v)$. A vector $x^* \in \partial f(x)$ iff*

$$f(y) - f(x) \geqslant \langle x^*, y - x\rangle, \quad \forall y \in C.$$

Definition 2.4. The *tangent cone* to C at x, denoted $T_C(x)$, is the set of all those $v \in X$ satisfying

$$d_C^\circ(x; v) = 0,$$

where $d_C^\circ(x; v)$ is the *distant function* of C, given by

$$d_C(x) = \inf\{\|x - c\| : c \in C\}.$$

Definition 2.5. The *normal cone* to C at x, denoted $N_C(x)$, is defined the polarity of its tangent cone

$$N_C(x) = (T_C(x))^\circ = \{x^* \in X^* : \langle x^*, v\rangle \leqslant 0, \quad \forall v \in T_C(x)\}.$$