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On computing of arbitrary positive integer powers for one type of even order tridiagonal matrices with eigenvalues on imaginary axis – II

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Abstract

This paper is an extension of the work [J. Rimas, On computing of arbitrary positive integer powers for one type of even order tridiagonal matrices with eigenvalues on imaginary axis – I, Appl. Math. Comput., in press] in which the general expression of the lth power ($l \in N$) for one type of even order tridiagonal matrices is given. In this new paper we present the complete derivation of this general expression. Expressions of eigenvectors of the matrix and of the transforming matrix and its inverse are given, too.

Keywords: Tridiagonal matrices; Eigenvalues; Eigenvectors; Chebyshev polynomials

1. Introduction

Solving some difference, differential equations and delay differential equations we meet the necessity to compute the arbitrary positive integer powers of square matrix [1,2]. In the work [3] the general expression of the lth power ($l \in N$) for one type of even order tridiagonal matrices with eigenvalues on the imaginary axis is presented. In this paper we give the complete derivation of the general expression, presented in [3]. Similar tridiagonal matrices with eigenvalues on the real axis were investigated in [4] and [5].

2. Derivation of general expression

Consider the nth order ($n = 2p,\ p \in N$) matrix B of the following type:

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$$B=\begin{pmatrix} 0 & 2 & & & & \\ -1 & 0 & 1 & & 0 & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ & 0 & & -1 & 0 & 1 \\ & & & & -2 & 0 \end{pmatrix}. \tag{1}$$

We will derive expression of the lth power ($l \in N$) of the matrix (1) applying the expression $B^l = TJ^lT^{-1}$ [6], where J is the Jordan's form of B, T is the transforming matrix. Matrices J and T can be found provided eigenvalues and eigenvectors of the matrix B are known. The eigenvalues of B are defined by the characteristic equation

$$|B-\lambda E|=0; \tag{2}$$

here E is the identity matrix of nth order.

In the paper [3] it is shown that the roots of the characteristic equation (2) (the eigenvalues of the matrix B) are defined by the expression

$$\lambda_k=-2\mathrm{i}\cos\frac{(k-1)\pi}{n-1},\quad k=1,2,3,\ldots,n \tag{3}$$

and the Jordan's form of the matrix B can be represented in the form

$$J=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n); \tag{4}$$

here i is the imaginary unity ($\mathrm{i}^2=-1$), $n=2p$ ($p \in N$) is an order of the matrix B (see (1)).

3. Eigenvectors of matrix B and transforming matrix T

Consider the relation $J=T^{-1}BT$ ($BT=TJ$); here B is the nth order matrix (1) ($n=2p,\ p\in N$), $J=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is the Jordan's form of B, $\lambda_u(u=\overline{1,n})$ are the eigenvalues of B, T is the transforming matrix. Since all the eigenvalues of B are simple eigenvalues, columns of the transforming matrix T are the eigenvectors of the matrix B [6]. Denoting vth column of T by $T_v(v=\overline{1,n})$, we have $T=(T_1\ T_2\ T_3\ \ldots\ T_n)$ and $(BT_1\ BT_2\ BT_3\ \ldots\ BT_n)=(T_1\lambda_1\ T_2\lambda_2\ T_3\lambda_3\ \ldots\ T_n\lambda_n)$. The latter expression gives

$$BT_k=T_k\lambda_k,\quad k=\overline{1,n}. \tag{5}$$

Solving the set of systems (5), we find the eigenvectors of the matrix B:

$$T_v=\begin{pmatrix} e(0)T_0\left(\frac{\mathrm{i}\lambda_v}{2}\right) \\ e(1)T_1\left(\frac{\mathrm{i}\lambda_v}{2}\right) \\ \vdots \\ e(n-1)T_{n-1}\left(\frac{\mathrm{i}\lambda_v}{2}\right) \end{pmatrix},\quad v=\overline{1,n}; \tag{6}$$

here

$$e(k)=\mathrm{e}^{-\mathrm{i}\frac{k\pi}{2}}\quad (k=0,1,2,3,\ldots), \tag{7}$$

$T_k(x)$ is the kth degree Chebyshev polynomial of the first kind:

$$T_k(x)=\cos k\arccos x,\quad |x|\leqslant 1. \tag{8}$$