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Type two computability of social choice functions and the Gibbard–Satterthwaite theorem in an infinite society

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Abstract

This paper investigates the computability problem of the Gibbard–Satterthwaite theorem [A.F. Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601; M.A. Satterthwaite, Strategyproofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory 10 (1975) 187–217] of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch [K. Weihrauch, A Simple Introduction to Computable Analysis, Informatik Berichte, vol. 171, second ed., Fern Universität Hagen, Hagen, 1995; K. Weihrauch, Computable Analysis, Springer-Verlag, 2000]. There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function in an infinite society. We will show that if there exists a dictator for a social choice function, it is computable in the sense of Type two computability, but if there exists no dictator it is not computable. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and his most preferred alternative is always chosen. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual.

Keywords: Type two computability; Social choice function; The Gibbard–Satterthwaite theorem; Infinite society; Strategy-proofness; Dictator

1. Introduction

This paper investigates the computability problem of the Gibbard–Satterthwaite theorem [6,12] of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch [20,21].[2] Arrow's impossibility theorem [1] shows that, with a finite number of individuals, for any social welfare function (binary social choice rule which satisfies some conditions) there exists a

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[2] In other papers, [15–18], we have studied the computability problems of some other theorems of social choice theory such as the Arrow impossibility theorem, the theorem by Duggan and Schwartz [3] about the existence of dictator for strategy-proof social choice correspondences and the strong candidate stability theorem by Dutta et al. [4].

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dictator. In contrast Refs. [5,7,8] show that in a society with an infinite number of individuals (an infinite society), there exists a social welfare function without dictator. On the other hand, about strategy-proof social choice functions, with a finite number of individuals, the Gibbard–Satterthwaite theorem [6,12] shows that there exists a dictator for any strategy-proof social choice function. In contrast Ref. [11] shows that in an infinite society, there exists a coalitionally strategy-proof social choice function without dictator.[3] A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and it chooses his most preferred alternative. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual.

In the next section we present the framework of this paper and some preliminary results. In Section 3 we will show the following results:

1. There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 1).
2. If there exists a dictator, the social choice function is computable in the sense of Type two computability, but if there exists no dictator it is not computable (Theorem 2).

A decisive set for a social choice function is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the social choice function does not choose y regardless of the preferences of other individuals.

Mihara [9] presented an analysis about the ordinary Turing machine computability of social choice rules. Since there are only countable number of ordinary Turing machines, he assumes that only countable number of profiles of individual preferences are observable. But Type two machine can treat uncountable input.

2. The framework and preliminary results

There are m ($\geqslant 3$) alternatives and a countably infinite number of individuals. m is a finite positive integer. The set of alternatives is denoted by A. The set of individuals is denoted by N. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive linear (strict) orders, that is, they prefer one alternative to another alternative, and are not indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Since there are a finite number of alternatives, the varieties of linear orders over the alternatives are finite. We denote the set of individual preferences by Σ. A combination of individual preferences, which is called a *profile*, is denoted by $p\ (= (\succ_1, \succ_2, \ldots))$, $p'\ (= (\succ'_1, \succ'_2, \ldots))$ and so on. The set of profiles is denoted by Σ^ω, where $\omega = \{1, 2, \ldots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a social choice function $f : \Sigma^\omega \to A$ which chooses at least one and at most one alternative corresponding to each profile of the revealed preferences of individuals. We require that social choice functions are *coalitionally strategy-proof*. This means that any group (coalition) of individuals cannot benefit by revealing preferences which are different from their true preferences, in other words, each coalition of individuals must have an incentive to reveal their true preferences, and cannot manipulate any social choice function. The coalitional strategy-proofness is an extension of the ordinary strategy-proofness which requires only non-manipulability by an individual. We also require that social choice functions are *onto*, that is, their ranges are A. The Gibbard–Satterthwaite theorem states that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function, or in other words there exists no social choice function which satisfies strategy-proofness and has no dictator. In contrast Ref. [11] shows that when the number of individuals in the society is infinite, there exists a coalitionally strategy-proof social choice function without dictator. A dictator of a social choice function is an individual whose most preferred alternative is always chosen by the social choice function.

[3] [19] is a recent book that discusses social choice problems in an infinite society.