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Multiple-soliton solutions for the Boussinesq equation

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Abstract

In this work we use the Hirota's direct method combined with the simplified version of this method to determine the N-soliton solutions for the Boussinesq equation. The one-soliton solutions will be handled by the tanh–coth method. The work highlights the significant features of the employed methods.

Keywords: Hirota's direct method; Boussinesq equation; tanh–coth method; Multiple-soliton solutions

1. Introduction

This paper is concerned with the multiple-soliton solutions of the fourth-order nonlinear Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{1}$$

with $u(x,t)$ is a sufficiently often differentiable function. Eq. (1) was introduced by Boussinesq to describe the propagation of long waves in shallow water [1–10]. It also arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma, and in vibrations in a nonlinear string. It is used in many physical applications such as the percolation of water in porous subsurface of a horizontal layer of material. This particular form (1) is of special interest because it is completely integrable and admits inverse scattering formalism.

A great deal of research work has been invested in recent years for the study of the Boussinesq equation. Ablowitz and Segur [5] implemented the inverse scattering transform method to handle this equation. Hirota [11–15] introduced the powerful bilinear formalism to handle the completely integrable nonlinear equations to establish the N-soliton solution. Nimmo et al. [6] introduced an alternative formulation of the N-soliton solutions in terms of some functions of the Wronskian determinant of N functions.

We aim from this work to use the Hirota's direct method developed in [11–15], and used thoroughly in [16–18]. The Hirota's method will be combined with the simplified version of this bilinear formalism introduced by Hereman et al. in [7,8] to determine N-soliton solutions for the Boussinesq equation. The tanh–coth

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method [19–24] will be employed to formally derive single soliton and periodic solutions. We also aim to emphasize the power of the proposed methods in effectively handling integrable nonlinear evolution equations.

2. The methods

For single soliton solutions, the tanh–coth method [19–24] is usually used among other simple methods. However, the Hirota bilinear formalism [11–15] and a simplified version of this method [7,8] will be used to address the concept of N-soliton solutions. In what follows we will present brief steps of the two methods, where details can be found in [5–22] and the references therein.

2.1. The tanh–coth method

A wave variable $\xi = (x - ct)$ converts a PDE to an ODE

$$Q(u, u', u'', u''' \cdots) = 0. \tag{2}$$

The last equation is integrated as long as all terms contain derivatives where integration constants are considered zeros.

The standard tanh method [19,20] introduces a new independent variable

$$Y = \tanh(\mu\xi), \quad \xi = x - ct, \tag{3}$$

that leads to the change of derivatives:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} &= \mu(1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y}, \\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} &= -2\mu^2 Y(1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y} + \mu^2(1 - Y^2)^2\frac{\mathrm{d}^2}{\mathrm{d}Y^2}. \end{aligned} \tag{4}$$

The tanh–coth method [21,22] admits the use of a finite expansion of tanh and coth functions

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{5}$$

where M is a positive integer that will be determined. Expansion (5) reduces to the standard tanh method for $b_k = 0,\ 1 \leqslant k \leqslant M$. Substituting (5) into the reduced ODE results in an algebraic equation in powers of Y. Balancing the linear terms of highest order in the resulting equation with the highest order nonlinear terms leads to the determination of the parameters M, a_k and b_k.

2.2. Hirota's bilinear method

A well-known method, called the direct method, developed by Hirota [8–13] is used for deriving N-soliton solutions for completely integrable equations. Hirota introduced the bilinear differential operators

$$D_t^m D_x^n(a \cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t)b(x',t')|_{x=x',t=t'}. \tag{6}$$

The solution of the Boussinesq equation can be expressed by

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f, \tag{7}$$

where $f(x,t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \tag{8}$$

where ϵ is a formal expansion parameter. For the one-soliton solution we set

$$f(x,t) = 1 + \epsilon f_1, \tag{9}$$