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The improvements of Chebyshev–Halley methods with fifth-order convergence

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Abstract

In this paper, we present a family of new fifth-order methods, which improves the classical Chebyshev–Halley methods with cubic convergence. The new methods add one evaluation of the function at the point iterated by Chebyshev–Halley methods. Analysis of efficiency shows that the new methods can be of practical interest, which is also corroborated by numerical tests.

Keywords: Chebyshev–Halley methods; Newton's method; Non-linear equations; Iterative method; Root-finding

1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

Newton's method (NM) for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is an important and basic method [1], which converges quadratically.

A family of third-order methods, called Chebyshev–Halley methods [2], is written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'(x_n)^2}.$$

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This family includes the classical Chebyshev's method (CM) ($\alpha = 0$), Halley's method (HM) ($\alpha = 1/2$) and Super-Halley method (SHM) ($\alpha = 1$) (for the details of these methods, see [3–6] or a recent review [7]).

In order to improve the local order of convergence, Grau and Díaz-Barrero [8] propose an improvement of Chebyshev's method with fifth-order convergence

$$\tilde{x}_{n+1} = x_n - \left(1 + \frac{f''(x_n)(f(x_n) + f(x_{n+1}))}{2f'(x_n)^2}\right)\frac{f(x_n) + f(x_{n+1})}{f'(x_n)}, \tag{3}$$

where x_{n+1} is defined by (2) ($\alpha = 0$), namely a Chebyshev's iterate.

Recently, by the composition of Newton and Chebyshev–Halley methods

$$\tilde{x}_{n+1} = x_{n+1} - \frac{f(x_{n+1})}{f'(x_{n+1})},$$

and Taylor approximation of $f'(x_{n+1})$

$$f'(x_{n+1}) \simeq f'(x_n) + f''(x_n)(x_{n+1} - x_n),$$

a family of fifth-order methods is obtained in [9]

$$\tilde{x}_{n+1} = x_{n+1} - \frac{f(x_{n+1})}{f'(x_n) + f''(x_n)(x_{n+1} - x_n)}, \tag{4}$$

where x_{n+1} is defined by (2).

These methods are very interesting because they can improve the order of convergence and computational efficiency of the classical third-order methods with an additional evaluation of the function.

In this paper, we modify the approach used in [9] by using Taylor approximation for the inverse function and derive a new family of fifth-order methods. Then analysis of convergence is supplied. Their practical utility is demonstrated by numerical results.

2. The methods

In [10], Newton's theorem for $y = f(x)$ is used to derive Newton's method and modified Newton's method with cubic convergence, while Homeier [11] uses it for the inverse function $x(y)$. Now, we use the approach of Homeier on a new interval of integration

$$x(y) = x(y_{n+1}) + \int_{y_{n+1}}^{y} x'(t)\,\mathrm{d}t. \tag{5}$$

We apply the rectangular rule to compute the integral of (5) and arrive at

$$x(y) = x(y_{n+1}) + x'(y_{n+1})(y - y_{n+1}). \tag{6}$$

Using $y = \tilde{y}_{n+1} = y(\tilde{x}_{n+1}) = 0$, $x(y) = \tilde{x}_{n+1}$, and $x(y_{n+1}) = x_{n+1}$ equivalent to $y_{n+1} = f(x_{n+1})$, we obtain the approximate formula

$$\tilde{x}_{n+1} = x_{n+1} - x'(y_{n+1})y_{n+1}. \tag{7}$$

Now, we will derive the approximation of $x'(y_{n+1})$. Since

$$x'(y) = f'(x(y))^{-1},$$

$$x''(y) = (x'(y))' = (f'(x(y))^{-1})' = -\frac{f''(x(y))}{f'(x(y))^3},$$

and using Taylor expansion for the first derivative $x'(y)$ of the inverse function $x(y)$ instead of $f'(x)$, we can approximate $x'(y_{n+1})$ as

$$x'(y_{n+1}) \simeq x'(y_n) + x''(y_n)(y_{n+1} - y_n) = \frac{1}{f'(x_n)} - \frac{f''(x_n)}{f'(x_n)^3}(f(x_{n+1}) - f(x_n)). \tag{8}$$