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A new type of weighted quadrature rules and its relation with orthogonal polynomials

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Dedicated to mathematicians who work on quadrature rules and orthogonal polynomials

Abstract

In this research, we introduce a new type of weighted quadrature rules as

$$\int_{\alpha}^{\beta} \rho(x)(f(x) - P_{m-1}(x;f))\,\mathrm{d}x = \sum_{i=1}^{n} a_{i,m} f^{(m)}(b_{i,m}) + R_n^{(m)}[f],$$

in which $P_{m-1}(x;f) = \sum_{j=0}^{m-1} f^{(j)}(\lambda)(x-\lambda)^j/j!$; $\lambda \in \mathbf{R}$; $m \in \mathbf{N}$; $\rho(x)$ is a positive function; $f^{(m)}(x)$ denotes the mth derivative of the function $f(x)$ and $R_n^{(m)}[f]$ is the error function. We determine the error function analytically and obtain the unknowns $\{a_{i,m}, b_{i,m}\}_{i=1}^{n}$ explicitly so that the above formula is exact for all polynomials of degree at most $2n+m-1$. In particular, we emphasize on the sub-case

$$\int_{\alpha}^{\beta} \rho(x)(f(x) - f(\lambda))\,\mathrm{d}x = \sum_{i=1}^{n} a_{i,1} f'(b_{i,1}) + R_n^{(1)}[f],$$

with the precision $2n$ (one degree higher than Gauss quadrature precision degree) and show that under some specific conditions the two foresaid formulas can be connected to the current weighted quadrature rules. The best application of the case $m = 1$ in the second formula is when λ is a known root of the function $f(x)$. For instance, $\int_{\alpha}^{\beta} \rho(x)(\int_{\lambda}^{x} g(t)\,\mathrm{d}t)\,\mathrm{d}x$ and $\int_{\alpha}^{\beta} \rho(x)(x-\lambda)g(x)\,\mathrm{d}x$ are two samples in which $f(\lambda) = 0$. Finally, we present various analytic examples of above rules and introduce a more general form of the mentioned formulas as

$$\int_{\alpha}^{\beta} \rho(x)(f(x) - P_{m-1}(x;f))\,\mathrm{d}x = \sum_{i=1}^{n} \sum_{j=0}^{k} d_i^{(m+j)} f^{(m+j)}(r_i) + R_n^{(m,k)}[f].$$

Keywords: Weighted quadrature rules; Precision degree; Undetermined coefficients method; Orthogonal polynomials; Weight function; Moments; Integral equation

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1. Introduction

Let us start our discussion with the general form of weighted quadrature rules [4,7,13], which is defined as

$$\int_a^b w(x)f(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i) + \sum_{k=1}^{m} v_k f(z_k) + E_n^{(m)}[f], \tag{1}$$

where $w(x)$ is a positive function on $[a,b]$; $\{w_i\}_{i=1}^n$, $\{v_k\}_{k=1}^m$ are unknown coefficients; $\{x_i\}_{i=1}^n$ are unknown nodes and finally $\{z_k\}_{k=1}^m$ are the predetermined nodes. The error function $E_n^{(m)}[f]$ is also determined (see e.g. [1,2,6]) by

$$E_n^{(m)}[f] = \frac{f^{(2n+m)}(\xi)}{(2n+m)!}\int_a^b w(x)\prod_{k=1}^{m}(x-z_k)\prod_{i=1}^{n}(x-x_i)^2\,\mathrm{d}x, \quad a<\xi<b. \tag{2}$$

In general, it is proved that the Gaussian quadrature (for $m=0$ in (1)) has the highest precision degree $(2n-1)$ among the other quadrature rules of this type. In other words, if the equality

$$\int_a^b w(x)f(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i) + E_n^{(0)}[f], \tag{3}$$

holds, then

$$E_n^{(0)}[f] = 0 \Longleftrightarrow f(x) \in \left\{\mathbf{\Pi}_k(x) = \sum_{r=0}^{k} c_r^{(k)} x^r\right\}_{k=0}^{k=2n-1}, \tag{4}$$

and the nodes $\{x_i\}_{i=1}^n$ in (3) are the zeros of a sequence of polynomials, say $P_n(x)$, which is orthogonal [3,14] with respect to the weight function $w(x)$ on $[a,b]$, i.e.

$$\int_a^b w(x)P_n(x)P_m(x)\,\mathrm{d}x = \left(\int_a^b w(x)P_n^2(x)\,\mathrm{d}x\right)\delta_{n,m}; \quad \delta_{n,m} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{5}$$

Moreover, to obtain the coefficients $\{w_i\}_{i=1}^n$ in (3) it is not necessary to solve the following linear system of order $n \times n$:

$$\sum_{i=1}^{n}(x_i^k)w_i = \int_a^b x^k w(x)\,\mathrm{d}x, \quad k = 0,1,\ldots,2n-1, \tag{6}$$

rather, one can straightforwardly use the following formulas [13]:

$$w_j = \frac{\langle P_{n-1}(x)|P_{n-1}(x)\rangle}{P_{n-1}(x_j)P_n'(x_j)}, \quad j = 1,2,\ldots,n, \tag{7}$$

where

$$\langle P_i(x)|P_j(x)\rangle = \int_a^b w(x)P_i(x)P_j(x)\,\mathrm{d}x, \tag{7.1}$$

and/or

$$\frac{1}{w_j} = \sum_{i=0}^{n-1} P_i^{*2}(x_j), \quad j = 1,2,\ldots,n, \tag{8}$$

where $P_i^*(x)$ are orthonormal polynomials of $P_i(x)$ defined by

$$P_i^*(x) = \frac{P_i(x)}{\langle P_i(x)|P_i(x)\rangle^{1/2}}. \tag{8.1}$$

A direct way to prove that the precision degree of (3) is $2n-1$ is to apply the Hermite interpolation formula [5,6]