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Convergence ball of a modified secant method with convergence order 1.839... ☆

Hongmin Ren [a,*], Qingbiao Wu [b]

[a] *College of Information and Engineering, Hangzhou Radio and TV University, Hangzhou 310012, Zhejiang, PR China*
[b] *Department of Mathematics, Zhejiang University, Hangzhou 310028, Zhejiang, PR China*

Abstract

A local convergence theorem on a modified secant method with convergence order 1.839... for solving nonlinear equations is established under the hypotheses that the second-order and third-order derivative of the function involved are bounded. An estimate of the radius of the convergence ball of this method is obtained. Moreover, an error estimate is provided which matches the convergence order of the method.

Keywords: Modified secant method; Nonlinear equation; Convergence ball; Error estimate

1. Introduction

In this study we are concerned with the convergence ball and error estimate of a modified secant method which is used to solve the following nonlinear equation

$$f(x) = 0, \tag{1}$$

where f is defined on an open domain or closed domain D on a real space $\mathbf{R}$ or complex space $\mathbf{C}$.

There are kinds of methods to find a solution of (1). Iterative methods are often used to solve this problem. If we use the famous Newton's method (see [1,2]), we can do as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n \geqslant 0) \ (x_0 \in D). \tag{2}$$

Under the reasonable hypotheses, Newton's method converges quadratically. But the computation of first-order derivative at each step is needed and in many cases it is not easy for us. To avoid this, we can use the secant method (see [1]) instead of as follows:

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* Corresponding author.
E-mail addresses: rhm@mail.hzrtvu.edu.cn (H. Ren), qbwu@zju.edu.cn (Q. Wu).

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$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]} \quad (n \geqslant 0) \;\; (x_{-1}, x_0 \in D), \tag{3}$$

where $f[.,.]$ is a divided difference of order one. Under reasonable conditions, the secant method has convergence order 1.618... Using the property of divided differences (see [3]), we have

$$f'(x_n) = f[x_n, x_n] = f[x_{n-1}, x_n] + (x_n - x_{n-1})f[x_{n-1}, x_n, x_n], \tag{4}$$

where $f[.,.,.]$ is a divided difference of order two. If we discard the second term of (4), i.e., we replace $f'(x_n)$ in Newton's method with $f[x_{n-1}, x_n]$, we can get the secant method (3) at once. To get a higher convergence order, we consider to replace $f'(x_n)$ in Newton's method with a better estimate. Because the computation of $f[x_{n-1}, x_n, x_n]$ in (4) needs the computation of the derivative, we replace it with $f[x_{n-2}, x_{n-1}, x_n]$. Now we obtain the following modified secant method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n] + (x_n - x_{n-1})f[x_{n-2}, x_{n-1}, x_n]} \quad (n \geqslant 0) \;\; (x_{-2}, x_{-1}, x_0 \in D). \tag{5}$$

The method (5) needs almost the same cost of computation as the secant method and has a higher convergence order than the secant method. Therefore, it attracts interest of many authors (see [4–6]). In this paper, a local convergence theorem on the method (5) is established by estimating the radius of its convergence ball.

Suppose $x_\star$ is a solution of (1). An open ball $B(x_\star, r)$ with center $x_\star$ and radius r is called a convergence ball of an iterative method, if the sequence generated by this iterative method starting from any initial points in it converges. Estimates of the radius of the convergence ball for Newton's method, the secant method, two-point Newton-like methods and a modified secant method have been given by Refs. [7–12]. Under the conditions $|f'(x_\star)^{-1}f''(x)| \leqslant M$ $(M > 0,\ x \in D)$, $|f'(x_\star)^{-1}f'''(x)| \leqslant N$ $(N > 0,\ x \in D)$, the radius of the convergence ball of the modified secant method (5) is proved to be $\frac{12}{\sqrt{81M^2+48N}+9M}$ at least in this work, an error analysis is given which shows the order of the method is 1.839... at least. Finally, two examples are provided to show the application of our theorem.

2. Convergence ball and error analysis

In this section, we give the convergence ball and error analysis of the modified secant method.

Theorem 1. *Suppose $x_\star$ is a solution of Eq. (1), $f'(x_\star) \neq 0$, $|f'(x_\star)^{-1}f''(x)| \leqslant M$ and $|f'(x_\star)^{-1}f'''(x)| \leqslant N$ hold for any $x \in D$, where $M > 0$ and $N > 0$ are constants. Denote $R = \frac{12}{\sqrt{81M^2+48N}+9M}$. Then the sequence $\{x_n\}$ generated by the modified secant method (5) starting from any three initial points $x_{-2}, x_{-1}, x_0 \in B(x_\star, R)$ is well defined, and converges to the unique solution $x_\star$ in $B(x_\star, \frac{2}{M}) \subseteq D$ that is bigger than $B(x_\star, R)$. Moreover, the following error estimate is satisfied*

$$|x_\star - x_n| \leqslant R\left(\frac{|x_\star - x_2|}{R}\right)^{F_{n-1}+F_{n-2}+F_{n-3}} \left(\frac{|x_\star - x_1|}{R}\right)^{F_{n-1}+F_{n-2}} \left(\frac{|x_\star - x_0|}{R}\right)^{F_{n-1}} \quad (n \geqslant 0), \tag{6}$$

where F_n is the Fibonacci generalized sequence, and is defined by $F_{-3} = 0$, $F_{-2} = -1$, $F_{-1} = 1$, $F_{n+1} = F_n + F_{n-1} + F_{n-2}$ $(n \geqslant -1)$.

Proof. Denote $B_n = f[x_{n-1}, x_n] + (x_n - x_{n-1})f[x_{n-2}, x_{n-1}, x_n]$ $(n \geqslant 0)$, $e_n = x_\star - x_n$ $(n \geqslant -2)$. Suppose x_{n-2}, x_{n-1} and x_n are well defined by the modified secant method (5) for a fixed integer n, and x_{n-2}, x_{n-1}, $x_n \in B(x_\star, R)$. From the conditions in this theorem, of course the above supposition is satisfied for $n = 0$.

Next we will show $B_n \neq 0$, and we consider to estimate $|1 - f'(x_\star)^{-1}B_n|$. By $x_{n-2}, x_{n-1}, x_n \in B(x_\star, R)$, $|f'(x_\star)^{-1}f''(x)| \leqslant M (x \in D)$, $|f'(x_\star)^{-1}f'''(x)| \leqslant N$ $(x \in D)$, $R = \frac{12}{\sqrt{81M^2+48N}+9M}$ and the property of divided differences, it follows