Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 188 (2007) 749–761
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Optimal *B*-spline collocation method for self-adjoint singularly perturbed boundary value problems

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Abstract

We present a *B*-spline collocation method of higher order for a class of self-adjoint singularly perturbed boundary value problems. The essential idea in this method is to divide the domain of the differential equation into three non-overlapping subdomains and solve the regular problems obtained by transforming the differential equation with respective boundary conditions on these subdomains using the present higher order *B*-spline collocation method. The boundary conditions at the transition points are obtained by the asymptotic approximation of order zero to the solution of the problem. The convergence analysis is given and the method is shown to have optimal order convergence; by collocating the perturbed differential equation, which is satisfied by a special cubic spline interpolate of the true solution. Numerical experiments are conducted to demonstrate the efficiency of the method.

Keywords: Singular perturbation problems; Boundary value problems; Domain decomposition; Asymptotic approximation; *B*-spline collocation method

1. Introduction

Consider the following class of self-adjoint singularly perturbed two point boundary value problems:

$$L_1 u \equiv -\varepsilon^2 u''(x) + b(x)u(x) = f(x) \quad \text{for } 0 \leqslant x \leqslant 1 \tag{1}$$

with boundary conditions

$$u(0) = \rho_1 \quad \text{and} \quad u(1) = \rho_2, \tag{2}$$

where ε is a small parameter and $f(x)$, $b(x)$ are smooth functions and satisfy $b(x) \geqslant b^* > 0$, $\forall x \in [0, 1]$ for some constant b^*. The boundary value problem (1) and (2) under these assumptions posses unique solution $u(x)$. In general, as ε tends to zero, the solution $u(x)$ may exhibit exponential boundary layers at both ends of the interval $[0, 1]$.

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.10.029

These problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier–Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction-diffusion process, quantum mechanics, optimal control etc. due to the variation in the width of the layer with respect to the small perturbation parameter ε. Several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Then the mesh needs to be refined substantially to grasp the solution within the boundary layers. To avoid this, a piecewise uniform mesh was first constructed by Shishkin [1]. Miller et al. [2] discussed the fitted numerical method with piecewise uniform mesh for singularly perturbed boundary value problems.

Several numerical methods have been developed for the numerical solution of self-adjoint singularly perturbed boundary value problems, in particular to the problems having the boundary layers at one or both ends of the interval. Boglaev [3], Schatz and Wahlbin [4] used the finite element technique to solve such types of problems. Miller [5] gave sufficient conditions for the first-order uniform convergence of three-point difference scheme. O'Riordan and Stynes in [6–9] introduced the concept of frozening the coefficients by considering the piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the coefficient terms $b(x)$ and $f(x)$ of singularly perturbed boundary value problem (1) and (2). While Stojanovic [10] gave an optimal difference scheme by considering the quadratic interpolating splines instead of piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the coefficient $f(x)$. Surla and Jerkovic [11] considered the spline collocation method for the solution of singularly perturbed boundary value problems. Kadalbajoo and Aggarwal [12] introduced the B-spline collocation method with fitted mesh technique for self-adjoint singularly perturbed boundary value problem and proved the second order uniform convergence of the method. Bawa and Natesan [13] gave a computational method with quintic spline difference scheme for self-adjoint singularly perturbed boundary value problem and showed the fourth-order convergence of the method. However, our present method gives optimal order convergence, that is, fourth order convergence with cubic splines.

This paper is arranged as follows. Some higher order cubic-spline interpolation results which are used in the construction of the higher order collocation method are discussed in Section 2. In Section 3, the formulation of higher order B-spline collocation method is given for two point boundary value problem. The B-spline collocation method is described in Section 4. In Section 5, we divide the whole domain of the differential equation into three non-overlapping subdomains and the regular problems are obtained by transforming the differential equation with respective boundary conditions on these subdomains and are solved by using higher order B-spline collocation method. The optimal order convergence of B-spline collocation method is given in Section 6. In Section 7, numerical experiments are conducted to demonstrate the efficiency of the proposed method. Results of experiments are discussed in Section 8 and finally the conclusions are included in Section 9.

2. Higher order interpolation relations

In this section we present some results on the cubic spline interpolation. These results are valid with specific choice of boundary conditions and needed in the analysis of the convergence rates.

Let $\Pi : a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$, where $x_{i+1} - x_i = h = (b-a)/N$, for $0 \leqslant i \leqslant N-1$, be a uniform partition of the interval $[a, b]$. The function $s(x)$ is said to be a cubic spline over Π if $s(x) \in C^2[a, b]$ and $s(x)$ restricted to $[x_{i-1}, x_i]$ is a cubic polynomial for $1 \leqslant i \leqslant N$. Let $S_3(\Pi)$ be the space of all such cubic splines.

Suppose $u(x)$ is a sufficiently smooth function defined on $[a, b]$ and $s(x)$ satisfies

$$s(x_i) = u(x_i) \quad \text{for } 0 \leqslant i \leqslant N, \tag{3}$$

then $s(x)$ is said to be an $S_3(\Pi)$ interpolate of $u(x)$. Since $\dim S_3(\Pi) = N + 3$, two additional linearly independent conditions are taken near the endpoints to determine an unique $S_3(\Pi)$-interpolate of $u(x)$. That is

$$s''(a) = u''(a) - \frac{h^2}{12} u''''(a) \quad \text{and} \quad s''(b) = u''(b) - \frac{h^2}{12} u''''(b). \tag{4}$$

The above interpolation problem given in (3) with two additional boundary conditions on $s''(a)$ and $s''(b)$ in (4) is well posed (see [14]). Thus $s(x)$ can be uniquely determined by $u(x)$ with (3) and (4).