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A numerical scheme for the variance of the solution of the random transport equation

Maria Cristina C. Cunha, Fábio Antonio Dorini *

Departamento de Matemática Aplicada, IMECC – UNICAMP, Cidade Universitária Zeferino Vaz, CP. 6065, 13083-970 Campinas SP, Brazil

Abstract

We present a numerical scheme, based on Godunov's method (REA algorithm), for the variance of the solution of the 1D random linear transport equation, with homogeneous random velocity and stochastic initial condition. We obtain the stability conditions of the method and we also show its consistency with a deterministic nonhomogeneous advective–diffusive equation, which means convergency. Numerical results are considered to validate our scheme.

Keywords: Random linear transport equation; Finite volume schemes; Godunov's method

1. Introduction

In this work we are concerned about the variance of the solution of the random transport equation,

$$\begin{cases} Q_t(x,t) + AQ_x(x,t) = 0, & t > 0,\ x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), \end{cases} \tag{1}$$

with a homogeneous random transport velocity, A, and stochastic initial condition, $Q_0(x)$. The solution, $Q(x,t)$, is a random function. For the particular case, Riemann problem (1) with

$$Q(x,0) = \begin{cases} Q_0^- & \text{if } x < 0, \\ Q_0^+ & \text{if } x > 0, \end{cases} \tag{2}$$

where Q_0^- and Q_0^+ are random variables, we presented in [1] the expression for the solution:

$$Q_R(x,t) = Q_0^- + X\left(Q_0^+ - Q_0^-\right), \tag{3}$$

where X is a Bernoulli random variable with $P(X=0) = 1 - F_A\left(\frac{x}{t}\right)$ and $P(X=1) = F_A\left(\frac{x}{t}\right)$; here $F_A(x)$ is the cumulative probability function of the random variable A.

* Corresponding author.
E-mail addresses: cunha@ime.unicamp.br (M.C.C. Cunha), fabio.dorini@gmail.com (F.A. Dorini).

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Also, according to [1], considering the independence between A and both Q_0^-, Q_0^+, the statistical mean and variance are given by

$$\langle Q_R(x,t)\rangle = \langle Q_0^-\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right] \tag{4}$$

and

$$\mathrm{Var}[Q_R(x,t)] = \mathrm{Var}[Q_0^-] + F_A\left(\frac{x}{t}\right)[\mathrm{Var}[Q_0^+] - \mathrm{Var}[Q_0^-]] + F_A\left(\frac{x}{t}\right)\left[1 - F_A\left(\frac{x}{t}\right)\right]\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right]^2. \tag{5}$$

In our point of view, the special attraction of (3)–(5) is their utilization in discretizations of stochastic equations, like (1). In [2] we present an explicit method to calculate the first statistical moment of $Q(x,t)$, the solution of (1) with $Q(x,0) = Q_0(x)$ a random function. In that report we show that the Godunov method provides a numerical scheme for the statistical mean which is, under certain assumptions on the discretization, stable and consistent with a diffusive equation. Therefore, besides the scheme itself, the numerical approach also gives an effective equation compatible with one published in the literature.

The aim of this paper is to improve the knowledge of the random solution of (1) with the random function $Q(x,0) = Q_0(x)$. We present a numerical method to calculate the variance of $Q(x,t)$, which is the quantity most commonly used to specify the dispersion of the distribution around its mean.

In Section 2 we deduce the explicit numerical scheme using the Godunov's ideas. Consistency, stability and convergency are analyzed in Section 3. Finally, in Section 4, we present some numerical examples.

2. The numerical scheme

In this section we present the numerical scheme for the variance of the solution of (1). We denote the spatial and the time grid points by $x_j = j\Delta x$ and $t_n = n\Delta t$, respectively, and the jth grid cell is $\mathscr{C}_j = (x_{j-1/2}, x_{j+1/2})$, $x_{j\pm 1/2} = x_j \pm \frac{\Delta x}{2}$. Let Q_j^n be an approximation of the cell average of $Q(x,t_n)$:

$$Q_j^n \simeq \frac{1}{\Delta x}\int_{\mathscr{C}_j} Q(x,t_n)\mathrm{d}x = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} Q(x,t_n)\mathrm{d}x. \tag{6}$$

Assuming that the cell averages at time t_n, Q_j^n, are known, we summarize the REA, for Reconstruct-Evolve-Average, algorithm [3,4] in three steps:

[Step 1.] Reconstruct a piecewise polynomial function, $\widetilde{Q}(x,t_n)$, from the cell averages Q_j^n. In our case we use the piecewise constant function with Q_j^n in the jth cell, i.e., $\widetilde{Q}(x,t_n) = Q_j^n$ for all $x \in \mathscr{C}_j$.

[Step 2.] Evolve the equation exactly, or approximately, with this initial data to obtain $\widetilde{Q}(x,t_{n+1})$ a time Δt later.

[Step 3.] Average $\widetilde{Q}(x,t_{n+1})$ over each grid cell to obtain the new cell averages, i.e.,

$$Q_j^{n+1} = \frac{1}{\Delta x}\int_{\mathscr{C}_j} \widetilde{Q}(x,t_{n+1})\mathrm{d}x.$$

At a time t_n, the piecewise constant function, step 1, defines a set of Riemann problems in each $x = x_{j-1/2}$: the differential equation (1) with the initial condition

$$Q(x,t_n) = \begin{cases} Q_{j-1}^n & \text{if } x < x_{j-1/2}, \\ Q_j^n & \text{if } x > x_{j-1/2}. \end{cases} \tag{7}$$

We may use (3) to find a local solution to each Riemann problem at a time $\frac{\Delta t}{2}$ later:

$$Q(x,t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right], \tag{8}$$

where, for a x sufficiently close to $x_{j-1/2}$, $X(x)$ is the Bernoulli random variable:

$$X(x) = \begin{cases} 1, & P(X(x) = 1) = F_A(x), \\ 0, & P(X(x) = 0) = 1 - F_A(x). \end{cases} \tag{9}$$