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Linearizability conditions for a cubic system

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Abstract

We obtain the necessary and sufficient conditions for linearizability of an eight-parameter family of two-dimensional system of differential equations in the form of linear canonical saddle perturbed by polynomials with four quadratic and four cubic terms.

Keywords: Isochronicity; Linearizability; Centers; Linear normal forms

1. Introduction

We consider a polynomial system of differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^{q} = P(x,y),\\ -\frac{\mathrm{d}y}{\mathrm{d}t} &= y - \sum_{(p,q)\in S} b_{qp}x^{q}y^{p+1} = -Q(x,y), \end{aligned} \tag{1}$$

where x, y, a_{pq}, b_{qp} are complex variables, $S = \{(p_m, q_m) | p_m + q_m \geqslant 1, m = 1, \ldots, l\}$ is a subset of $\{-1 \cup \mathbb{N}\} \times \mathbb{N}$, and $\mathbb{N}$ is the set of non-negative integers. The notation in (1) simply emphasizes that we take into account only nonzero coefficients of the polynomials. By (a, b) we will denote the ordered vector of the coefficients of system (1), $(a,b) = (a_{p_1q_1}, \ldots, a_{p_lq_l}, b_{q_lp_l}, \ldots, b_{q_1p_1})$. In the case when

$$x = \bar{y}, \quad a_{ij} = \bar{b}_{ji}, \quad \mathrm{i}\mathrm{d}t = \mathrm{d}\tau \tag{2}$$

(the bar stands for the complex conjugate numbers), the system (1) is equivalent to the equation

$$\mathrm{i}\frac{\mathrm{d}x}{\mathrm{d}\tau} = x - \sum_{(p,q)\in S} a_{pq}x^{p+1}\bar{x}^{q}, \tag{3}$$

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which has a center or a focus at the origin in the real plane $\{(u, v)|x = u + \mathrm{i}v\}$, where the system can be also written in the form

$$\dot{u} = -v + U(u, v), \quad \dot{v} = u + V(u, v). \tag{4}$$

For systems of the form (4) (where the power series expansions of U and V at the origin start with quadratic terms) the notions of center and isochronicity have a simple geometric meaning. Namely, the origin of system (4) is a *center* if all trajectories in its neighborhood are closed and it is an *isochronous* center if the period of oscillations is the same for all these trajectories. According to the Poincaré–Lyapunov Theorem system (4) has a center at the origin if and only if it admits a first integral of the form

$$\Psi(u, v) = u^2 + v^2 + \sum_{k+m=3}^{\infty} \theta_{km} u^k v^m.$$

Since after the complexification $x = u + \mathrm{i}v$, $y = \bar{x}$ the above integral has the form

$$\Psi(x, y) = xy + \sum_{k+m=3}^{\infty} \eta_{km} x^k y^m, \tag{5}$$

following to Dulac we say that system (1) has a center at the origin if it admits a first integral of the form (5).

We recall (see, for instance, [1,6] for the details) that the real analytical system (4) has an isochronous center if and only if it can be transformed to the linear system

$$\dot{u} = v, \quad \dot{v} = -u,$$

that is, if the normal form of (4) is linear. Thus for system (4) the notion of isochronicity is equivalent to the notion of linearizability. However the problem of linearizability arises also for more general system (1), which in the special case when the conditions (2) hold is equivalent to (3) and, therefore, to (4). In this paper we will study the problem of linearizability for system (1). Namely, we will consider the problem how to decide if a polynomial system (1) can be transformed to the linear system

$$\dot{z}_1 = z_1, \quad \dot{z}_2 = -z_2 \tag{6}$$

by means of a formal change of the phase variables

$$z_1 = x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a, b) x^m y^j, \quad z_2 = y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a, b) x^m y^j. \tag{7}$$

If for some values of the parameters a_{pq}, b_{qp} such transformation exists we say that the corresponding system (1) is *linearizable* (or has a *linearizable center* at the origin). It follows from a result of Poincaré and Lyapunov that if there is a formal transformation (7) linearizing (1) then it converges in a neighborhood of the origin.

Although the study of isochronicity goes back at least to Huygens who investigated the oscillations of cycloidal pendulum, at present the problem is of renewed interest. In particular, in recent years many studies has been devoted to the investigation of the linearizability (isochronicity) problem for different subfamilies of the cubic system (that is, the system (1) where the right hand sides are the polynomials of degree three, see, e.g. [2–4,6,8,7,10] and references therein). In this paper we study the problem of linearizability for the following eight-parameter cubic system:

$$\begin{aligned} \dot{x} &= x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{02}y^2), \\ \dot{y} &= -y(1 - b_{10}x - b_{01}y - b_{02}y^2 - b_{20}x^2). \end{aligned} \tag{8}$$

2. Preliminaries

Taking derivatives with respect to t in both parts of each of the equalities in (7), we obtain

$$\dot{z}_1 = \dot{x} + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(m x^{m-1} y^j \dot{x} + j x^m y^{j-1} \dot{y}),$$

$$\dot{z}_2 = \dot{y} + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(m x^{m-1} y^j \dot{x} + j x^m y^{j-1} \dot{y}).$$