Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 189 (2007) 126–130
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

A one-parameter family of third-order methods to solve nonlinear equations

Changbum Chun

School of Liberal Arts, Korea University of Technology & Education, 307 ByongCheon-Myun, Cheonan City, Chungnam 330-708, Republic of Korea

Abstract

In this paper we present a new one-parameter family of iterative methods to solve nonlinear equations which includes some well-known third-order methods as particular ones. The convergence analysis shows that the order of convergence of each method of the family is three. Numerical examples are given to illustrate the efficiency and performance of the presented methods.

Keywords: Iterative methods; Nonlinear equations; Newton's method; Root-finding; Order of convergence

1. Introduction

One of the most basic problems in mathematics is that of solving nonlinear equation

$$f(x) = 0. \tag{1}$$

To solve Eq. (1), we can use iterative methods such as Newton's method and its variants [1,2]. The Newton's method is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0. \tag{2}$$

Let us consider the geometric construction of Newton's method: given an iterate x_n, we consider the linear equation

$$y = ax + b \tag{3}$$

and then determine the unknowns a and b by imposing the tangency conditions

$$y(x_n) = f(x_n), \quad y'(x_n) = f'(x_n), \tag{4}$$

E-mail address: cbchun@kut.ac.kr

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.11.058

thereby obtaining the tangent line

$$y(x) = f(x_n) + f'(x_n)(x - x_n) \tag{5}$$

to the graph of f at $(x_n, f(x_n))$. The zero of this tangent line is x_{n+1}, the next iterate defined by (2).

The well-known third-order methods which entail the evaluation of f'' can be obtained by admitting geometric derivation from the quadratic curves such as parabola and hyperbola.

If we consider the parabola

$$x^2 + ax + by + c = 0 \tag{6}$$

and we impose the tangency conditions

$$y(x_n) = f(x_n), \quad y'(x_n) = f'(x_n), \quad y''(x_n) = f''(x_n), \tag{7}$$

we have

$$y(x) - f(x_n) = f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2. \tag{8}$$

The point x_{n+1} where the graph of (8) intersects with x-axis gives us the iterative process called irrational Halley's method [2,3]

$$x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L(x_n)}} \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0, \tag{9}$$

where

$$L(x) = \frac{f(x)f''(x)}{f'^2(x)}. \tag{10}$$

If we consider the parabolas

$$ay^2 + y + bx + c = 0 \tag{11}$$

and the hyperbola

$$axy + y + bx + c = 0 \tag{12}$$

and the tangency conditions (7) are imposed to (11) and (12) respectively, then taking the points of intersection of the curves with x-axis as next iterates yield the iteration scheme called Chebyshev's method [2]

$$x_{n+1} = x_n - \left(1 + \frac{L(x_n)}{2}\right) \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0 \tag{13}$$

and the scheme called Halley's method [4]

$$x_{n+1} = x_n - \left(\frac{2}{2 - L(x_n)}\right) \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0, \tag{14}$$

respectively.

Sharma's recent geometric derivation of a third-order family of methods is noteworthy [5]. It consists in considering the quadratic equation in x and y of the form

$$x^2 + ay^2 + bx + cy + d = 0. \tag{15}$$

Taking into account that the quadratic equation (15) passes through the point $(x_n, y(x_n))$, (15) can be written in the equivalent form

$$Q(x, y) = (x - x_n)^2 + a_n(y - y(x_n))^2 + b_n(x - x_n) + c_n(y - y(x_n)) + d_n = 0. \tag{16}$$

If we impose the same tangency conditions as (7) to determine the values of b_n, c_n and d_n in terms of a_n, we have

$$b_n = \frac{2(1 + a_n f'^2(x_n))f'(x_n)}{f''(x_n)}, \tag{17}$$