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Matrix inverse problem and its optimal approximation problem for R-symmetric matrices ☆

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Abstract

Let $R \in C^{n\times n}$ be a nontrivial involution, i.e., $R^2 = I$ and $R \neq \pm I$. A matrix $A \in C^{n\times n}$ is called R-symmetric if $RAR = A$. The solvability conditions and the expression of the matrix inverse problem for R-symmetric matrices with $R^* = R$ are derived, also the least-squares solutions of the matrix inverse problem for R-symmetric matrices with $R^* = R$ are given. The corresponding optimal approximation problem for R-symmetric matrices with $R^* = R$ is considered. We firstly point out that the optimal approximation problem is solvable, then get the expression of its unique solution. It can be seen that this paper generalizes the results mentioned in Zhou [F.-Z. Zhou, L. Zhang, X.-Y. Hu, Least-square solutions for inverse problem of centrosymmetric matrices, Comput. Math. Appl. 45 (2003) 1581–1589].

Keywords: R-symmetric matrix; Matrix inverse problem; Least-squares solution; Optimal approximation problem

1. Introduction

In this paper the following notations are considered and used. Let $C^{m\times n}$ and $C_r^{m\times n}$ denote the sets of all $m\times n$ complex matrices and all $m\times n$ complex matrices with rank r. $UC^{n\times n}$ denotes the set of $n\times n$ complex unitary matrices. I_k denotes the identity matrix of order k. We denote by A^* and A^+ Hermitian matrix and Moore–Penrose generalized inverse of matrix A, respectively. Let $\|A\|$ be the Frobenius norm of matrix A. For matrices $A, b \in C^{n\times m}$, let $\langle A, b\rangle = \mathrm{tr}(B^*A)$ denote the inner product of matrices A and B. Therefore, $C^{n\times m}$ is a Hilbert inner product space and the norm of a matrix generated by the inner product is the Frobenius norm.

Let $R \in C^{n\times n}$ be a nontrivial involution, i.e., $R^2 = I$ and $R \neq \pm I$. A matrix $A \in C^{n\times n}$ is called R-symmetric if $RAR = A$. We denote by $RSC^{m\times n}$ the set of all $m\times n$ complex R-symmetric matrices. In particular, let J be the flip matrix with ones on the secondary diagonal and zeros elsewhere, then a J-symmetric matrix A is

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centrosymmetric, i.e., $a_{n-i+1,n-j+1} = a_{i,j}, 1 \leqslant i,j \leqslant n$. Centrosymmetric matrices have been extensively studied, we refer readers to [1–7]. Generalizations of centrosymmetric matrices have been studied recently in [8–16]. More and more people are absorbed in the research on R-symmetric matrices and some results [17–20] on R-symmetric matrices are derived.

In this paper, we will consider the matrix inverse problem and the optimal approximation problem for R-symmetric matrices,which can be described as follows.

Problem 1 (*Matrix Inverse Problem*). Given $X, b \in C^{n\times m}$, find a matrix $A \in RSC^{n\times n}$ such that

$$AX = B. \tag{1}$$

The solvability conditions and the expression of the solution of Problem 1 for R-symmetric matrices with $R^* = R$ are derived. However, the matrices X and B occurring in practice are usually obtained from experiments, it is difficult for them to satisfy the solvability conditions,so it is necessary to consider the least-squares solutions of the inverse problem.

Problem 2. Given $X, b \in C^{n\times m}$, find a matrix $\widetilde{A} \in RSC^{n\times n}$ such that

$$\|\widetilde{A}X - B\| = \min_{A\in RSC^{n\times n}} \|AX - B\|. \tag{2}$$

We will give the expression of the solution of Problem 2. Furthermore we will consider the optimal approximation problem associated with $AX = B$ for R-symmetric matrix. That is

Problem 3 (*Optimal Approximation Problem*). Given $D \in C^{n\times n}$, then find $A_D \in S_E$ such that

$$\|D - A_D\| = \min_{A\in S_E} \|D - A\|, \tag{3}$$

where S_E is the solution set of Problem 1 or Problem 2.

We also will give the expression of the solution of Problem 3.

This paper is organized as follows. In Section 2, we will firstly give the expression of the least-squares solution of Problem 2, then get the solvability conditions and the expression of the solution of Problem 1. In Section 3, we will point out that Problem 3 is solvable and the expression of the unique solution of Problem 3 is derived. By the way, this paper generalized the results given in [15].

2. The solutions of Problems 1 and 2

In this section, we will first give the expression of the least-squares solution of Problem 2, then the solvability conditions and the expression of the solution of Problem 1.

Firstly, some notions and results on R-symmetric matrices will be needed.

If λ is an eigenvalue of $A \in C^{n\times n}$, Let $\varepsilon_A(\lambda)$ denote the λ-eigenspace of A. A vector $z \in C^n$ is R-symmetric if $Rz = z$ or R-skew symmetric if $Rz = -z$,then $\varepsilon_R(1)$ and $\varepsilon_R(-1)$ are the subspaces of $C^{n\times n}$ consisting of R-symmetric and R-skew symmetric vectors, respectively. An eigenvalue λ of A is said to be R-even if $\varepsilon_A(\lambda)$ contains a nonzero R-symmetric vector or R-odd if $\varepsilon_A(\lambda)$ contains a nonzero R-skew symmetric vector. Let $r = \dim[\varepsilon_R(1)]$ and $s = \dim[\varepsilon_R(-1)]$. Since an involution is diagonalizable and $R \neq \pm I, r, s \geqslant 1$ and $r + s = n$. Let $\{p_1 \cdots p_r\}$ and $\{q_1 \cdots q_s\}$ be orthonormal bases for $\dim[\varepsilon_R(1)]$ and $\dim[\varepsilon_R(-1)]$ respectively, and define

$$P = [p_1 \cdots p_r] \in C^{n\times r}, \quad Q = [q_1 \cdots q_s] \in C^{n\times s}. \tag{4}$$

Although P and Q are not unique, finding suitable P and Q are straightforward. In the worse case the columns of P and Q could be found by applying the Gram–Schmidt process to the columns of $I + R$ and $I - R$, respectively. If R is a signed permutation matrix this needs little computation. For example,

if $R = J_{2m}$, we can take

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m \\ J_m \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m \\ -J_m \end{bmatrix}, \tag{5}$$