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An efficient algorithm for solving Troesch's problem ☆

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Abstract

A new algorithm is presented for solving Troesch's problem. The numerical scheme based on the modified homotopy perturbation technique is deduced. Some numerical experiments are made. Compared with the variational iteration method and the Adomian decomposition method, the scheme is shown to be highly accurate, and only a few terms are required to obtain accurate computable solutions. Finally, the algorithm is applied to other problems.

Keywords: Troesch's problem; Homotopy perturbation technique; Taylor expansion; Variational iteration method; Adomian decomposition method

1. Introduction

In this paper, we consider the boundary-value problem, Troesch's problem [1–3]

$$u'' = \lambda \sinh(\lambda u), \quad 0 \leqslant x \leqslant 1 \tag{1}$$

with the boundary conditions $u(0) = 0$, $u(1) = 1$.

Troesch's problem was described and solved by Weibel [4]. It arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure. This problem has been studied extensively. Troesch found its numerical solution by the shooting method [5]. A numerical algorithm based on the decomposition technique is presented by Deeba et al. [6], but the procedure is more complex. Though Momani et al. [7] applied the variational iteration method to Troesch's problem, the numerical results is not effective. The closed form solution to this problem in terms of the Jacobian elliptic function has been given [3] as

$$u(x) = \frac{2}{\lambda}\sinh^{-1}\left\{\frac{u'(0)}{2} sc\left(\lambda x | 1 - \frac{1}{4}u'^2(0)\right)\right\}, \tag{2}$$

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where $u'(0)$, the derivative of u at 0, is given by the expression $u'(0) = 2\sqrt{1-m}$, with m being the solution of the transcendental equation

$$\frac{\sinh\left(\frac{\lambda}{2}\right)}{\sqrt{1-m}} = sc(\lambda|m), \tag{3}$$

where the Jacobian elliptic function $sc(\lambda|m)$ [1,2] is defined by $sc(\lambda|m) = \frac{\sin\phi}{\cos\phi}$, where ϕ, λ and m are related by the integral

$$\lambda = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2\theta}}\mathrm{d}\theta.$$

From (2), it was noticed [3,6] that a pole of $u(t)$ occurs at a pole of $sc\left(\lambda x|1 - \frac{1}{4}u'(0)^2\right)$. It was also noticed that the pole occurs at

$$x \approx \frac{1}{2\lambda}\ln\left(\frac{16}{1-m}\right).$$

It also has an equivalent definition given in terms of a lattice.

At present, we also have other methods for solving two-point boundary-value problems. For instance, the finite difference method [8–10], the homotopy method [11], the homotopy analysis method [12–14], the homotopy perturbation method [12–19], etc. Based on the modified homotopy perturbation technique, we introduce a small parameter and use Taylor expansion to study the problem in this paper. The algorithm is implemented for three special cases of the problem. Compared with the Adomian decomposition method [6] and the variational iteration method [7], the numerical results show that the scheme approximates the exact solution with a high degree of accuracy, using only few terms of the numerical algorithm. And the procedure is more simple.

An outline of the paper is as follows. In Section 2, the modified homotopy perturbation technique will be presented as it applies to Troesch's problem. In Section 3, the algorithm is implemented for three numerical examples. In the last section, the algorithm is applied to other problems.

2. Modified homotopy perturbation technique

We modify the homotopy perturbation technique [13–16] and directly introduce a parameter $p \in [0, 1]$ in Troesch's equation in the form

$$\begin{cases} u'' = \lambda\sinh(\lambda p u), & \text{(a)} \\ u(0) = 0, \quad u(1) = 1. & \text{(b)} \end{cases} \tag{4}$$

It is obvious that when $p = 0$, Eqs. (4a) and (4b) becomes a linear equation; when $p = 1$, it becomes the original nonlinear one.

Due to the fact that $p \in [0, 1]$, so the imbedding parameter can be considered as a "small parameter". Applying the perturbation technique [20,21], we can assume that the solution of Eqs. (4a) and (4b) can be expressed as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots. \tag{5}$$

Setting $p = 1$ results in the approximate solution of Eq. (1):

$$u^* = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{6}$$

To obtain its approximate solution of Eqs. (4a) and (4b), we expand sinh (λu) around $\hat{u}$

$$\sinh(\lambda u) = \sinh(\lambda\hat{u}) + \lambda\cosh(\lambda\hat{u})(u-\hat{u}) + \frac{\lambda^2\sinh(\lambda\hat{u})}{2!}(u-\hat{u})^2 + \frac{\lambda^3\cosh(\lambda\hat{u})}{3!}(u-\hat{u})^3 + \frac{\lambda^4\sinh(\lambda\hat{u})}{4!}(u-\hat{u})^4 + \cdots.$$