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A convergent quadratic-time lattice algorithm for pricing European-style Asian options ☆

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Abstract

Asian options are strongly path-dependent derivatives. Although efficient numerical methods and approximate closed-form formulas are available, most lack convergence guarantees. Asian options can also be priced on the lattice. All efficient lattice algorithms keep only a polynomial number of states and use interpolation to compensate for the less than full representation of the states. Let the time to maturity be partitioned into n periods. This paper presents the first $O(n^2)$-time convergent lattice algorithm for pricing European-style Asian options; it is the most efficient lattice algorithm with convergence guarantees. The algorithm relies on the Lagrange multipliers to choose optimally the number of states for each node of the lattice. The algorithm is also memory efficient. Extensive numerical experiments and comparison with existing PDE, analytical, and lattice methods confirm the performance claims and the competitiveness of our algorithm. This result places the problem of European-style Asian option pricing in the same complexity class as that of the vanilla option on the lattice.

Keywords: Option pricing; Binomial model; Trinomial model; Path-dependent derivative; Asian option; Complexity; Lagrange multiplier; PDE; Lattice

1. Introduction

Path-dependent derivatives have payoffs that depend strongly on the price history of the underlying asset. In pricing such derivatives, the historical information needs to be encoded as part of the state. Although the effect tends to enlarge the state space, it may not lead to exponential complexity if done properly. For example, barrier and look back options can be efficiently priced despite the fact that they are path-dependent. For other

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path-dependent derivatives, however, the performance issue is more intricate. The Asian option is perhaps the most representative of them.

Asian options are originally traded in the Asian markets, particularly Tokyo [1]. The payoff of the Asian option depends on the arithmetic average price of the underlying asset. It is therefore useful for hedging transactions whose cost is related to the average price of the underlying asset. The price of the Asian option is also less subject to price manipulation. Hence the averaging feature is popular in many thinly traded markets and embedded in complex derivatives such as the refix clauses in convertible bonds. Asian option is first suggested by [2].

Pricing Asian options has been a practical and research problem of long standing when the underlying asset's price is lognormally distributed. The source of the difficulty lies in that the sum of lognormal random variables is no longer lognormally distributed. Several solutions have been proposed for the problem, including analytical approximations, Monte Carlo simulation, lattices, and partial differential equations (PDEs).

Approximate closed-form formulas have been derived under various assumptions. These formulas have been evaluated thoroughly in [1,3–5]. The general conclusion is that all are as good as their assumptions, and most lack convergence guarantees. For example, some formulas lose accuracy for low volatility levels, whereas others do so for high volatility levels. Lower and upper bounds on the option price are derived in [6–9]. As no simple, exact closed-form solutions exist yet, the development of efficient numerical algorithms becomes an important alternative. First, there are the popular Monte Carlo and the related quasi-Monte Carlo methods surveyed in [10]. Both the Monte Carlo approach and the analytical approach suffer from the inability to handle early exercise without bias. Although Longstaff and Schwartz have developed a least-squares Monte Carlo approach to tackle the problem [11], a convergence proof remains elusive (see [12] for a convergence proof in the case of American-style vanilla options). Other drawbacks of Monte Carlo include its probabilistic nature and relative inefficiency.

The third type of approach, the lattice and the closely related PDE methods, are more general as they can handle early exercise. The main challenge with the lattice method in the case of Asian options is its exponential nature: An exponential number of arithmetic operations seem needed for an exact evaluation. This is because every price path, which corresponds to a state (that is, the average to date, also called the running average), leads to a different average price, thus payoff as well. To reduce the complexity, all known practical lattice algorithms keep only a small subset of the states. When an option value for a missing state is called for in the pricing algorithms, it is interpolated from the option values of the neighboring states. This successful paradigm is due to Hull and White [13] and Ritchken et al. [14] and is followed by, for example, Zvan et al. [15] and Klassen [16]. We will call it the interpolation paradigm. The interpolation paradigm obviously introduces interpolation error, and a major concern is whether the magnitude of the interpolation error converges to zero. Pricing Asian options with two-dimensional PDEs also tackles the issue of exploding state space with the interpolation paradigm [17].

Partition the time to maturity into n periods. It is well-known that a binomial lattice with n periods contains about $n^{2/2}$ nodes (see Fig. 1). Observe that the binomial model has a lattice structure because it recombines. Let the average number of states (running averages) kept at each node be k, a critical adjustable parameter for lattice algorithms. As the total number of states is $kn^2/2$, the asymptotic running time of the lattice algorithm is $O(kn^2)$. An algorithm must decide upon how to distribute these $kn^2/2$ states among the nodes. The choice

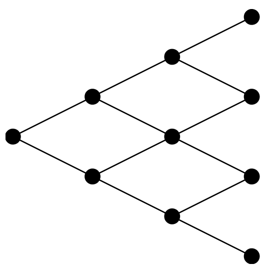

Fig. 1. Binomial lattice. Each node has two successor nodes. The number of nodes at any time i is $i+1$. The total number of nodes of an n-period binomial lattice is $\sum_{i=0}^{n}(i+1) = (n+2)(n+1)/2 \approx n^2/2$.