Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 189 (2007) 2004–2010
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Modified efficient variant of super-Halley method

Arif Rafiq [a,*], Muhammad Awais [a], Fiza Zafar [b]

[a] *Department of Mathematics, COMSATS Institute of Information Technology, Plot No. 30, Sector H-8/1, Islamabad 44000, Pakistan*
[b] *Centre for Advanced Studies in Pure and Applied Mathematics, B.Z. University, Multan 60800, Pakistan*

Abstract

In this paper, we present a new modified variant of super-Halley method for solving non-linear equations. Numerical results show that the method has definite practical utility.

Keywords: Iterative method; Two-step methods; Three-step methods; Nonlinear equations; Numerical examples

1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D, is a scalar function.

The following algorithms exist in the literature [1,4,5].

The classical Newton's method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \tag{1.2}$$

This is an important and basic method [1], which converges quadratically.

An acceleration of Newton's method, called super-Halley method, is introduced in [2,3], written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \tag{1.3}$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}.$$

* Corresponding author.
E-mail addresses: arafiq@comsats.edu.pk (A. Rafiq), awais.maths@gmail.com (M. Awais), fizazafar@gmail.com (F. Zafar).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.12.086

The following method was introduced by Kou et al. [4]:

$$\begin{aligned} x_{n+1} &= x_n - \left(1 + \frac{1}{2}\frac{\overline{L_f}(x_n)}{1-\overline{L_f}(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \\ \overline{L_f}(x_n) &= \frac{f''(x_n - \theta\frac{f(x_n)}{f'(x_n)})f(x_n)}{f'^2(x_n)}, \quad \theta \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Remark 1. For $\theta = 0$, we get the super-Halley method (1.3).

Kou et al. [4] also proved the following theorem in [4].

Theorem 2. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the neighborhood of the root α, then the order of convergence of the method defined by (1.4) is four if $\theta = 1/3$.*

It may be mentioned here that, for $\theta = 1/3$, (1.4) reduces to the following method:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\widetilde{L}_f(x_n)}{1-\widetilde{L}_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \tag{1.5}$$

where

$$\widetilde{L}_f(x_n) = \frac{f''\left(x_n - \frac{f(x_n)}{3f'(x_n)}\right)f(x_n)}{f'^2(x_n)}.$$

Kou et al. have developed the following two-step method in [5] as follows:

$$\begin{aligned} y_n &= x_n + \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)}. \end{aligned} \tag{1.6}$$

From a practical point of view, it is interesting to investigate higher-order variants of super-Halley method for general non-linear equations. In this paper, we present a modified three-step variant of super-Halley method with fourth-order convergence. A detailed convergence analysis of the new method is supplied. Consequently, the new method turned out to be of practical interest, as we show by some examples.

The method is

$$\begin{aligned} z_n &= x_n + \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \\ y_n &= z_n - \theta\frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \left(1 + \frac{1}{2}\frac{\widetilde{L}_f(x_n)}{1-\widetilde{L}_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1.7}$$

where

$$\widetilde{L}_f(x_n) = \frac{f''(y_n)f(x_n)}{f'^2(x_n)}.$$

2. Analysis of convergence

In this section, we compute the convergence order of the proposed method (1.7).