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Cubic spline polynomial for non-linear singular two-point boundary value problems

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Abstract

We present a cubic spline polynomial for the solution of non-linear singular two-point boundary value problems. The quesilinearization technique is used to reduce the non-linear problem to a sequence of linear problems. The resulting sets of differential equations are modified at the singular point and are treated by using cubic spline for finding the numerical solution. The method is tested on three physical model problems from the literature.

Keywords: Two-point boundary value problems; Cubic spline; Quesilinearization; Singular point

1. Introduction

We consider a class of non-linear singular two-point boundary value problem

$$y''(x) + \frac{\alpha}{x}y'(x) = f(x,y), \tag{1}$$

$$y'(0) = 0, \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where $\alpha \geqslant 1$ and β is a constant. We assume that $f(x,y)$ is continuous function, and $\frac{\partial f}{\partial y}$ exists, is continuous and $\frac{\partial f}{\partial y} \geqslant 0$. Due to the singularity at $x = 0$ on the left side of the differential equation (1), direct numerical techniques face convergence difficulties. Attempts by many researchers for the removal of singularity are based on using the series expansion procedures in the neighborhood $(0, \delta)$ of singularity (δ is vicinity of the singularity) and then solve the regular boundary value problem in the interval $(\delta, 1)$ using any numerical method. A three-point finite difference scheme has been proposed by Russell and Shampine [6] with Newton's iteration procedure for solving non-linear singular problems. A shooting algorithm based on a Taylor series method is developed by Rentrop [5]. The application of fourth order finite difference method to these problems has been discussed by Chawla et al. [2].

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In this paper, we discuss a direct method based on cubic spline approximation [1] for the solution of non-linear singular two-point boundary value problems. An advantage of the method is that the coefficient matrix of the system is of the system of Hessenberg form. First we use the quesilinearization technique to reduce the given non-linear problem to a sequence of linear problems. The resulting sets of differential equations are modified at the singular point and are treated by using cubic spline for finding the numerical solution. The numerical method is tested for its efficiency by considering three physical model problems from the literature.

2. Method of solution

The quesilinearization technique, has been used to reduce the given non-linear problem (1)–(3) to a sequence of linear problems. We choose a reasonable initial approximation for the function $y(x)$ in $f(x,y)$, call it as $y^{(0)}(x)$ and expand $f(x,y)$ around the function $y^{(0)}(x)$, we obtain

$$f\left(x,y^{(1)}\right)=f\left(x,y^{(0)}\right)+\left(y^{(1)}-y^{(0)}\right)\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(0)})}+\cdots \tag{6}$$

or in general we can write for $k=0,1,2,\ldots$ ($k=$ iteration index)

$$f\left(x,y^{(k+1)}\right)=f\left(x,y^{(k)}\right)+\left(y^{(k+1)}-y^{(k)}\right)\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(k)})}+\cdots \tag{7}$$

Eq. (1) can be

$$y_{xx}^{(k+1)}(x)+\frac{\alpha}{x}y_{x}^{(k+1)}(x)+a^{(k)}(x)y^{(k+1)}(x)=b^{(k)}(x),\quad k=0,1,2,\ldots, \tag{8}$$

where

$$a^{(k)}(x)=-\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(k)})},\quad b^{(k)}(x)=f(x,y^{(k)})-y^{(k)}\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(k)})}$$

subject to the boundary conditions

$$y_{x}^{(K+1)}(0)=0, \tag{9}$$

$$y^{(k+1)}(1)=\beta. \tag{10}$$

To solve the set of linear singular boundary value problems given by (8) subject to the boundary conditions (9) and (10). We first modify Eq. (8) at the singular point $x=0$ and then apply the cubic spline method. Following [4] by L'Hospital rule we transform the given problem (8)–(10) into

$$y_{xx}^{(k+1)}(x)+p^{(k)}(x)y_{x}^{(k+1)}(x)+q^{(k)}(x)y^{(k+1)}(x)=r^{(k)}(x),\quad k=0,1,2,\ldots, \tag{11}$$

$$y_{x}^{(K+1)}(0)=0, \tag{12}$$

$$y^{(k+1)}(1)=\beta, \tag{13}$$

where

$$p^{(k)}(x)=\begin{cases}0, & (x=0)\\ \frac{\alpha}{x}, & (x\neq 0)\end{cases},\quad q^{(k)}(x)=\begin{cases}\frac{a^{(k)}(0)}{\alpha+1}, & (x=0)\\ a^{(k)}(x), & (x\neq 0)\end{cases}$$

and

$$r^{(k)}(x)=\begin{cases}\frac{b^{(k)}(0)}{\alpha+1}, & (x=0)\\ b^{(k)}(x), & (x\neq 0)\end{cases}.$$

The range of the independent variable x is $[0,1]$. We choose points $0=x_0,x_1,\ldots,x_n=1$. Starting at x_0 a function $y^{(k+1)}(x)$ ($k=$ iteration index) is represented in $[x_0,x_1]$ by

$$S^{(k+1)}(x)=a+b(x-x_0)+c(x-x_0)^2+d_0(x-x_0)^3,\quad k=0,1,\ldots$$