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Analytic solutions for a class of differential equation with delays depending on state

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Abstract

In this paper, we study existence of analytic solutions of a class of the differential equation with state-dependent delays

$$x'(z)=\sum_{j=0}^{k}\sum_{t=1}^{\infty}C_{t,j}(z)(x^{[j]}(z))^t+G(z)$$

in the complex field $\mathbb{C}$. By constructing a convergent power series solution $y(z)$ of an auxiliary equation of the form

$$\alpha y'(\alpha z)=y'(z)\left[\sum_{j=0}^{k}\sum_{t=1}^{\infty}C_{t,j}(y(z))(y(\alpha^j z))^t+G(y(z))\right]$$

with $\alpha=G(0)\in\mathbb{C}\setminus\{0\}$, invertible analytic solutions of the form $y(\alpha y^{-1}(z))$ for the original equation are obtained.

Keywords: Differential equation with state-dependent delays; Analytic solution; Resonance; Diophantine condition; Brjuno condition

1. Introduction and hypotheses

Delay differential equation of the form

$$x'(z)=f(z,x(z),x(z-\tau_1(z)),\dots,x(z-\tau_k(z)))\tag{1.1}$$

have been lucubrated in [1,2]. However, such equations, when the delay functions $\tau_j(z)$ $(j=0,1,\dots,k)$ depend not only on the argument of the unknown function, but also state, $\tau_j(z)=\tau_j(z,x(z))$, have been relatively little researched. In this paper, we will discuss the existence of invertible analytic solutions to a class of functional differential equation of the form

$$x'(z)=\sum_{j=0}^{k}\sum_{t=1}^{\infty}C_{t,j}(z)(x^{[j]}(z))^t+G(z)\tag{1.2}$$

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in the complex field, where $x^{[j]}(z)$ denotes the jth iterate of $x(z)$. The above equation is a special case of (1.1), in which taking $f(z,y_1,\dots,y_k)=\sum_{t=1}^{\infty}C_{t,0}(z)z^t+\sum_{j=1}^{k}\sum_{t=1}^{\infty}C_{t,j}(z)y_j^t+G(z)$ and $\tau_j(z)=z-x^{[j-1]}(z)$. A distinctive feature of Eq. (1.2) is to include the sum of infinitely many terms in contrast to our previous considered equations [3,4,5,6,7].

Throughout this paper, we will assume that

(H1) Functions $C_{t,j}(z)$ $(t\in\mathbb{N}, j=0,1,\dots,k)$ and $G(z)$ are all analytic in $|z|<\sigma$ $(\sigma>0)$, and for each $j=0,1,\dots,k$, the series $\sum_{t=1}^{\infty}C_{t,j}(z_1)z_2^t$ converges for a definite pair of nonzero complex z_1, z_2 with $|z_1|<\sigma$.

(H2) $G(0)=\alpha\in\mathbb{C}\setminus\{0\}$.

As our previous works [3,4,5,6,7], we still reduce Eq. (1.2) with $x(z)=y(\alpha y^{-1}(z))$, called the Schröder transformation sometimes, to the auxiliary equation

$$\alpha y'(\alpha z)=y'(z)\left[\sum_{j=0}^{k}\sum_{t=1}^{\infty}C_{t,j}(y(z))(y(\alpha^j z))^t+G(y(z))\right]. \tag{1.3}$$

By constructing a convergent power series solutions $y(z)$ of Eq. (1.3), invertible analytic solutions of the form $y(\alpha y^{-1}(z))$ for Eq. (1.2) are obtained. Known in [3,4,5,6,7], the existence of analytic solutions for such equations is closely related to the position of α in the complex plane. In this paper, we will distinguish three different cases on α:

(**C1**) $0<|\alpha|<1$;
(**C2**) $\alpha=e^{2\pi i\theta}$, $\theta\in\mathbb{R}\setminus\mathbb{Q}$ and θ is a Brjuno number [8,9]: $B(\theta)=\sum_{n=0}^{\infty}\frac{\log q_{n+1}}{q_n}<\infty$, where $\{p_n/q_n\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ;
(**C3**) $\alpha=e^{2\pi i q/p}$ for some integer $p\in\mathbb{N}$ with $p\geqslant 2$ and $q\in\mathbb{Z}\setminus\{0\}$, and $\alpha\neq e^{2\pi i\xi/v}$ for all $1\leqslant v\leqslant p-1$ and $\xi\in\mathbb{Z}\setminus\{0\}$.

We observe that α is inside the unit circle S^1 in case (**C1**) but on S^1 in the rest cases. More difficulties are encountered for α on S^1 since the small divisor α^n-1 is involved in the latter (2.4). Under Diophantine condition: "$\alpha=e^{2\pi i\theta}$, *where* $\theta\in\mathbb{R}\setminus\mathbb{Q}$ *and there exist constants* $\zeta>0$ *and* $\delta>0$ *such that* $|\alpha^n-1|\geqslant\zeta^{-1}n^{-\delta}$ *for all* $n\geqslant 1$," the number $\alpha\in S^1$ is "far" from all roots of the unity and was considered in different settings [3,4,5,6,7]. Since then, we have been striving to give a result of analytic solutions for those α "near" a root of the unity, i.e., neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (**C2**) provides such a chance for us. Moreover, we also discuss the so-called the resonance case, i.e. the case of (**C3**).

2. Auxiliary equation in cases (C1) and (C2)

In this section, we discuss local invertible analytic solutions of Eq. (1.3) with the initial condition

$$y(0)=0,\quad y'(0)=\gamma\neq 0,\quad \gamma\in\mathbb{C}. \tag{2.1}$$

In order to study the existence of analytic solutions of (1.3) under the Brjuno condition, we first recall briefly the definition of Brjuno numbers and some basic facts. As stated in [10], for a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\}=\theta-[\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction

$$\theta=d_0+\theta_0=d_0+\frac{1}{d_1+\theta_1}=\cdots,$$

denoted simply by $\theta=[d_0,d_1,\dots,d_n,\dots]$, where d_j's and θ_j's are calculated by the algorithm: (a) $d_0=[\theta]$, $\theta_0=\{\theta\}$, and (b) $d_n=\left[\frac{1}{\theta_{n-1}}\right]$, $\theta_n=\left\{\frac{1}{\theta_{n-1}}\right\}$ for all $n\geqslant 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2}=1,\quad q_{-1}=0,\quad q_n=d_nq_{n-1}+q_{n-2},$$
$$p_{-2}=0,\quad p_{-1}=1,\quad p_n=d_np_{n-1}+p_{n-2}.$$

It is easy to show that $p_n/q_n=[d_0,d_1,\dots,d_n]$. Thus, for every $\theta\in\mathbb{R}\setminus\mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta)=\sum_{n\geqslant 0}\frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it satisfies Brjuno