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ScienceDirect
Applied Mathematics and Computation 186 (2007) 357–364
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

# Gauss quadrature rules for Cauchy principal value integrals with wavelets

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**Abstract**

Gauss quadrature rules for Cauchy principal value integrals with wavelets are established. By the construction of orthogonal polynomials of wavelets and scaling functions as weight functions, the wavelet Gauss quadrature formulae for Cauchy principal value integrals are introduced. Numerical examples are also presented.


*Keywords:* Wavelets; Gauss quadrature rules; Cauchy principal value integral

## 1. Introduction

In recent years, much progress has been made in establishing wavelet analysis as a competitive tool for image analysis/compression and for the numerical treatment of operator equations. Nevertheless, there are still some important problems which have not been solved satisfactorily yet. In this paper, we give a contribution to one of these questions as we shall now explain. The numerical approximation of integrals containing strongly singular integrals, in particular Cauchy principal value integrals, with wavelets or with the associated scaling functions, is a major issue when wavelet analysis is used to boundary integral approach for many types of differential equations. The methods connected to Cauchy principal value integrals have been addressed very intensively (see, [1–3] and references therein). However, this usually leads to serious trouble whenever these methods are used directly to the integrals with wavelets or with scaling functions. First of all, neither the wavelets nor the scaling functions are necessarily very smooth so that a classical quadrature rule may not perform satisfactorily. Moreover, in many cases, these functions are not know explicitly but only via certain functional equations from which the function value have to be computed or approximated. This is possible in principle, however, these kinds of functions evaluations may be expensive and/or inaccurate.

So far, several approached to this problem have been suggested. Using scaling functions as weight functions, Sweldens and Piessens [4,5] obtained the Newton–Cotes type quadrature formulae for wavelet integrals. Gauss type quadrature rules, owing to their optimization to nodes collocation, can get the optimal

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0096-3003/$ - see front matter 
doi:10.1016/j.amc.2006.07.106

degree of exactness. Therefore, many attentions have been paid to construct Gauss quadrature rules for wavelet integrals. The first approach in this direction was given by Gautschi et al. [6]. However their approach only works for positive refinable functions with some smoothness. In order to construct Gauss quadrature formulae for wavelets and scaling functions, which do not satisfy the non-negative condition of weight function, a "lifting trick" is used by Barinka et al. [7,8] and finally a general class of wavelet Gauss quadrature rules are obtained.

Motivated by the ideas mentioned above, orthogonal polynomials have been constructed with wavelets or scaling functions as weight function. And finally, Gauss quadrature rules for Cauchy principal value integrals with wavelets or scaling functions are derived. By far, to the best of our knowledge, no Gauss type quadrature rules are available for Cauchy principal value integrals with wavelets or scaling functions. This paper is organized as follows. In Section 2, some of the basic properties of wavelets related to this paper are briefly introduced. In Section 3, the construction of Gauss quadrature rules for wavelet and scaling function integrals are discussed in detail. Then Gauss quadrature rule for Cauchy principal value integrals with wavelets or scaling functions are obtained in Section 4. In Section 5, the results of the applications of these quadrature rules to some numerical examples are shown. Finally, a concise conclusion is given in Section 6.

## 2. Wavelet functions

Wavelets are usually constructed by means of a multiresolution analysis (MRA) introduced by Mallat. The fundamental theory of MRA and wavelet analysis can be found in many references, e.g. [9–11]. We shall briefly recall the basic properties of wavelets as far as it is needed for our purpose.

Let $\psi(x)$ and $\varphi(x)$ be the wavelet and the associated scaling function respectively. Their two-scale relations are given by

$$\varphi(x) = \sum_{j\in Z} h_j \varphi(2x-j), \tag{1}$$

$$\psi(x) = \sum_{j\in Z} g_j \varphi(2x-j), \tag{2}$$

where $h_j$s and $g_j$s satisfy $\sum_j h_j = 2, g_l = (-1)^j h_{1-j}$. For compactly supported wavelet and scaling function, we have

$$\text{Supp}(h_j) = \{j \in Z | h_j \neq 0\} \subseteq [m_1, m_2], \tag{3}$$

$$\text{Supp}(g_j) = \{j \in Z | g_j \neq 0\} \subseteq [n_1, n_2]. \tag{4}$$

Moreover, the scaling function satisfies

$$\int_{-\infty}^{+\infty} \varphi(x) = 1, \quad \sum_{j\in Z} \varphi(x-j) = 1. \tag{5}$$

## 3. Gauss quadrature formula for wavelets weight functions

The classical Gauss quadrature formula is

$$\int_b^a f(x)\omega(x)\,dx \approx I_\omega^n(f) = \sum_{i=1}^n \tau_i f(x_i), \tag{6}$$

where knots $x_i$ and weights $\tau_i$ are determined by the weight function $\omega(x)$, which is in general only required to be non-negative.

Defining the inner product corresponding to $\omega(\cdot) \geqslant 0$ as

$$(f,g)_\omega = \int_b^a f(x)g(x)\omega(x)\,dx \tag{7}$$