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Improving AOR method for consistent linear systems ☆

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Abstract

Milaszewicz [J.P. Milaszewicz, Improving Jacobi and Gauss–Seidel iterations, Linear Algebra Appl. 93 (1987) 161–170] and Gunawardena et al. [A.D. Gunawardena, S.K. Jain, L. Snyder, Modified iteration methods for consistent linear systems, Linear Algebra Appl. 154–156 (1991) 123–143] presented preconditioned methods for linear system in order to improve the convergence rates of Jacobi and Gauss–Seidel iterative schemes, respectively. In this paper, we apply AOR and SOR iterative schemes to the preconditioned linear systems and provide two theorems to improve the convergence rates of these iterative methods. Meanwhile, the methods in this paper can be used to more class of linear systems.

Keywords: Preconditioner; *L*-matrix; AOR method; SOR method

1. Introduction

Consider the following linear system:

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown. For any splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative method for solving (1) is

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, \ldots$$

For simplicity, without loss of generality, we assume throughout this paper that

$$A = I - L - U, \tag{2}$$

where I is the identity matrix, L and U are strictly lower and upper triangular matrices obtained from A, respectively. Then the iteration matrices of the classical AOR iterative method in [1] is defined:

$$L_{rw} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU], \tag{3}$$

where w and r are real parameters with $w \neq 0$.

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The spectral radius of the iterative matrix is decisive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. In order to accelerate the convergence of iterative method solving the linear system (1), preconditioned methods are often used. That is,

$$PAx = Pb,$$

where P, called the preconditioner, is a non-singular matrix. Let

$$PA = D^* - L^* - U^*.$$

Applying the AOR method, we get the corresponding preconditioned AOR iterative methods whose iterative matrices are

$$L^*_{rw} = (D^* - rL^*)^{-1}[(1-w)D^* + (w-r)L^* + wU^*],$$

where w and r are real parameters with $w \neq 0$.

In [4], Milaszewicz presented a new modified Jacobi and a modified Gauss–Seidel iterative method by using the preconditioner $P = I + \widetilde{S}$, where

$$\widetilde{S} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ -a_{31} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & 0 & \cdots & 0 \end{bmatrix}.$$

The author suggests that if the original iteration matrix is non-negative and irreducible, then performing Gaussian elimination on a selected column of iteration matrix to make it zero will improve the convergence of the iteration matrix.

In [5], Gunawardena et al. considered as a preconditioner $P = I + \overline{S}$, where

$$\overline{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

whose effect on A is to eliminate the elements of the first upper diagonal to improve the convergence of the iteration matrix.

In this paper, the preconditioned AOR-type iterative method for solving consistent linear system is presented. We apply AOR and SOR iterative schemes to the preconditioned linear systems and provide two theorems to improve the convergence rates of these iterative methods and generalized the results of [4,5].

2. Preparatory knowledge

For convenience, we shall now briefly explain some of the terminology and lemmas used in the next sections. Let $C = (c_{ij}) \in R^{n\times n}$ be an $n \times n$ real matrix. By diag(C), we denote the $n \times n$ diagonal matrix coinciding in its diagonal with c_{ii}. For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times n}$, we write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ holds for all $i, j = 1, 2, \ldots, n$. Calling A non-negative if $A \geqslant 0, (a_{ij} \geqslant 0;\ i, j = 1, \ldots, n)$, we say that $A - B \geqslant 0$ if and only if $A \geqslant B$. These definitions carry immediately over to vectors by identifying them with $n \times 1$ matrices. $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 1 [2]. A matrix A is a L-matrix if $a_{ii} > 0$; $i = 1, \ldots, n$ and $a_{ij} \leqslant 0$, for all $i, j = 1, 2, \ldots, n$; $i \neq j$.