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Preconditioners for elliptic problems via non-uniform meshes ☆

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Abstract

In this paper, we concerned with the preconditioned iterative methods for the solution of elliptic problems with Dirichlet or Robbins boundary condition via non-uniform meshes over an unit domain. We construct and analyze several preconditioners for the coefficient matrix derived from the finite difference discretization via non-uniform meshes. We prove that the preconditioners can be chosen so that the condition number of the preconditioned system can be reduced from $O(n^2)$ to $O(1)$. The analysis is supported by numerical experiments.

Keywords: Elliptic PDEs; Dirichlet boundary condition; Robbins boundary condition; Non-uniform mesh; PCG; Condition number

1. Introduction

In this paper, we concerned with the numerical solution of the elliptic problems with different boundary conditions on one-dimensional or two-dimensional unit domain by iterative method. After discretization by using standard finite-difference method and uniform or non-uniform meshes, such boundary value problem usually reduce to a large linear system of the form

$$Au = b, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is the coefficient matrix with fixed structure depending on the dimension and is usually symmetric, large and sparse when n is large. Here n is the number of the mesh points. In practice, such large linear systems are usually solved by iterative methods. One of the most popular iterative methods for solving such systems is the conjugate gradient (CG) method, see Axelsson and Barker [1]. At each step of the CG method, it is only need to evaluate the product of A with a known vector. Therefore, such methods are thought to be favorable with the sparsity which A possesses. Let u_k be the approximate solution obtained by the CG method after k steps, then it is well known that the estimate on convergence (see [2]) is as follows:

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$$\|u_k - u^*\| \leqslant 2\left(\frac{\sqrt{\mathscr{K}(A)}-1}{\sqrt{\mathscr{K}(A)}+1}\right)^k \|u_0 - u^*\|, \tag{2}$$

where u^* is the exact solution, u_0 is a given initial guess and $\mathscr{K}(A) \geqslant 1$ is the condition number of the coefficient matrix A. From this result, we can see that the convergence rate of the CG method depends on the condition number $\mathscr{K}(A)$. The smaller $\mathscr{K}(A)$ is, the faster the convergence of the CG method will be. In addition, some numerical examples show that, even if the condition number is large, the CG method will also be applied well if the spectrum of A are clustered in a small interval. Unfortunately, for elliptic problems such as (1), the eigenvalues of A usually are not clustered and usually the condition number is of $O(n^2)$, i.e.

$$\mathscr{K}(A) = O(h^{-2}) = O(n^2) \tag{3}$$

and hence this fact shows that the convergence of the CG method will be very slow. However, Facing these ill-posed problems, we can apply a preconditioned conjugate gradient (PCG) method, i.e. in stead of solving the original system (1), we will solve a preconditioned system

$$P^{-1}Au = P^{-1}b, \tag{4}$$

where the matrix P is called a preconditioner for the coefficient matrix A. It is chosen following two criteria: $Pr = d$ is easy to be solved for any arbitrary vector r; the spectrum of $P^{-1}A$ is clustered and/or the condition number of $P^{-1}A$ is much smaller than A (see [1]).

One of the successful classes of preconditioners for elliptic problems is the class of modified incomplete LU (ILU, MILU) factorizations, see for examples, Axelsson and Barker [1] and Dupont et al. [3]. It has been proved in Dupont et al. [3] that the condition numbers of the preconditioned systems for the ILU and the MILU methods are bounded by $O(n^2)$ and $O(n)$, respectively. Besides the ILU-type preconditioners, incomplete block Cholesky factorizations (INV, MINV) are another popular class of block preconditioners for solving large linear system (1). Numerical experiments in Concus et al. [4] point out that the condition numbers of the preconditioned system by INV and the MINV methods are bounded by $O(n^2)$ and $O(n)$, respectively.

In [5,6], Pickering and Harley present a fast Fourier transform (FFT) method for the solution of Poisson problem on a rectangular domain with Robbins boundary conditions on either one or two sides of the domain, combined with suitable conditions on the rest of the boundary. They proved that the number of iterations tends to be a constant value for some Robbins boundary conditions. In [7], Chan and Chan propose another class of preconditioners which is based on the fast Fourier transform (FFTs) for the numerical solutions of the second-order elliptic problems with Dirichlet boundary conditions. They proved that circulant preconditioners can be chosen so that the condition number of a preconditioned system is of $O(n)$. By similar approach, in [8], Huckle propose skew circulant preconditioners for the discretization coefficient matrix A. Another class of fast transform based preconditioners to the original elliptic problem are proposed by Chan and Wong [9]. They use sine transform based preconditioners for the coefficient matrix A. For rectangular domain, the condition number of the sine transform based preconditioned system is proved to be of $O(1)$. In [10], Ho and Ng construct transform based preconditioners [11] for the elliptic problems with Robbins boundary conditions. They proved that the condition number of the preconditioned system is of $O(1)$.

These precondition methods mentioned above are usually presented when we use a uniform meshes over the unit domains. Our purpose is to extend the preconditioners for a large linear system, which is obtained from the finite difference discretization via non-uniform meshes of the elliptic problems with different boundary conditions

$$A_h u = b_h, \tag{5}$$

where $A_h \in \mathbb{R}^{n\times n}$ is the coefficient matrix with fixed structure depending on the non-uniform meshes and is usually symmetric, large and sparse when n is large. The condition number of A is of $O(n^2)$, hence, when apply the CG method to the system (5), the convergence rate of the CG method is very slow. To reduce the condition number of A, we construct preconditioners for the elliptic problems with different boundary conditions. Furthermore, we will analyze the spectrum of the preconditioned system $P^{-1}A$ and show that the condition number of this preconditioned system is of $O(1)$, Thus P is effective and powerful.