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Matrix Krylov subspace methods for large scale model reduction problems

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Abstract

The present paper considers matrix Krylov subspace methods for solving large coupled Lyapunov matrix equations of the form $AP + PA^T + BB^T = 0$ and $A^TQ + QA + C^TC = 0$ where A is a real $n \times n$ matrix, B and C^T are real $n \times s$ and $n \times r$ matrices, respectively, with $s \ll n$ and $r \ll n$. Such problems appear in many areas of control theory such as the computation of the controllability and observability Gramians of a stable Linear Time Invariant (LTI) system. The proposed methods are based on the global Arnoldi and Lanczos processes. In the second part, we show how to use matrix Krylov subspace techniques to obtain a reduced order model for LTI systems. This will be done by approximating the corresponding transfer functions. Finally, numerical experiments are reported to illustrate the behavior and the effectiveness of the global Arnoldi and Lanczos processes when applied to solve some large coupled Lyapunov equations and to approximate some transfer functions.

Keywords: Coupled Lyapunov matrix equations; Global Arnoldi; Global Lanczos; Matrix Krylov subspaces; Transfer functions

1. Introduction

In this paper, we introduce new methods for large coupled Lyapunov matrix equations with low rank right-hand sides. These equations have the form

$$\begin{cases} AP + PA^T + BB^T = 0, \\ A^TQ + QA + C^TC = 0, \end{cases} \tag{1.1}$$

where A is an $n \times n$ real, large and sparse matrix ($n > 1000$), $P, Q \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times s}$, $C^T \in \mathbb{R}^{n \times r}$ with $\text{rank}(B) = s$, $\text{rank}(C) = r$ and $s \ll n$, $r \ll n$.

Each equation of (1.1) is a simple Lyapunov equation which arise frequently in different branches of engineering such as H_∞ optimal control theory [4] or stability analysis of dynamical systems [15]. Coupled

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equations of the form (1.1) appear also in the computation of the controllability P and observability Q Gramians of stable Linear Time Invariant (LTI) systems [16,21] having the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1.2}$$

in which $x(t)$ is the state vector of dimension n, $u(t)$ is the input or control vector of length s and $y(t)$ is the output vector of length r. The associated transfer function is given by $F(z) = C(zI_n - A)^{-1}B$.

Direct methods for the solution of Lyapunov matrix equations, such as those proposed in [2,6], are based on the Schur decomposition of A. As their complexity is $O(n^3)$, these methods are attractive if the matrix A is of a moderate size. In [13] a projection method based on the global Arnoldi process [11] was proposed to solve the Lyapunov equation

$$AX + XA^{\mathrm{T}} + GG^{\mathrm{T}} = 0, \tag{1.3}$$

where A is an $n \times n$ real, large and sparse matrix ($n > 1000$), $X \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times s}$ with $\mathrm{rank}(G) = s$ and $s \ll n$. Under the assumption that A is stable, it is well known that the above equation has a unique symmetric positive semi-definite solution X.

Based on the ideas given in [13], we consider iterative projection methods onto some matrix Krylov subspaces to generate low rank approximate solutions to coupled Lyapunov equations. These methods are based on the global Arnoldi process [11,13] and on the global Lanczos process [12]. These two processes allow us to obtain low dimensional Lyapunov equations that are solved by standard methods such as those given in [2,5,6]. Such a technique has been first proposed in [19] for Lyapunov equations with rank one right-hand side. In [9], the classical block Arnoldi process has been used to solve Lyapunov equations of the form (1.3). In the present work, we give a new upper bound for the residual norm associated to the approximate solution given by the Lyapunov global Arnoldi algorithm [13]. By approximating the transfer function $F(z)$ using the global full orthogonalization method [11], we also show how to apply the global Arnoldi process to obtain a reduced order model to (1.2).

The remainder of the paper is organized as follows. Section 2 is devoted to a short review of the global Arnoldi and global Lanczos processes. In Section 3, we recall how to extract low-rank approximate solutions to Lyapunov matrix equations by using the global Arnoldi algorithm and give a new upper bound for the residual norm. We also show how to use the global Arnoldi and global Lanczos processes to obtain new methods for the solution of the coupled Lyapunov equation (1.1). Section 4 considers the problem of approximating LTI systems by using matrix Krylov subspace techniques. The last section is devoted to some numerical experiments.

We use the following notations. For X and Y two matrices in $\mathbb{R}^{n\times s}$, we consider the following inner product $\langle X, Y\rangle_{\mathrm{F}} = \mathrm{tr}(X^{\mathrm{T}}Y)$, where $\mathrm{tr}(\cdot)$ denotes the trace. The associated norm is the Frobenius norm denoted by $\|\cdot\|_{\mathrm{F}}$. The notation $X \perp_{\mathrm{F}} Y$ means that $\langle X, Y\rangle_{\mathrm{F}} = 0$. For $V = [v_{i,j}] \in \mathbb{R}^{n\times s}$, we denote by $\mathrm{vec}(V)$ the vector of $\mathbb{R}^{n\times s}$ whose components are $v_{1,1}, \ldots, v_{n,1}; \ldots; v_{1,n}, \ldots, v_{n,n}$. Finally, $A \otimes B = [a_{i,j}B]$ denotes the Kronecker product of the matrices A and B. For this product, we use the following properties:

$$(A \otimes B)(C \otimes D) = (AC \otimes BD),$$

$$(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}},$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},$$

if A and B are invertible.

2. The global Arnoldi and Lanczos algorithms

These two processes were recently used in the context of iterative methods for large sparse matrix equations. Both processes use matrix-vector multiplications, hence they are very efficient for large sparse linear systems with multiple right-hand sides and related problems.