Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 181 (2006) 1423–1430
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

A new stable variable mesh method for 1-D non-linear parabolic partial differential equations

Urvashi Arora [a], Samir Karaa [b], R.K. Mohanty [c,*]

[a] Department of Mathematics, Rajdhani College, University of Delhi, New Delhi 110 015, India
[b] Department of Mathematics and Statistics, Sultan Qaboos University, Al-khod 123, Muscat, Oman
[c] Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110 007, India

Abstract

We propose a new stable variable mesh implicit difference method for the solution of non-linear parabolic equation $u_{xx} = \phi(x, t, u, u_x, u_t)$, $0 < x < 1$, $t > 0$ subject to appropriate initial and Dirichlet boundary conditions prescribed. We require only $(3 + 3)$-spatial grid points and two evaluations of the function ϕ. The proposed method is directly applicable to solve parabolic equation having a singularity at $x = 0$. The proposed method when applied to a linear diffusion equation is shown to be unconditionally stable. The numerical tests are performed to demonstrate the convergence of the proposed new method.

Keywords: Finite difference method; Implicit method; Variable mesh; Arithmetic average discretization; Diffusion equation; Burgers' equation

1. Introduction

We consider the non-linear parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \phi\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \quad 0 < x < 1, \;\; t > 0. \tag{1}$$

The initial condition is given by

$$u(x, 0) = f(x), \quad 0 \leqslant x \leqslant 1 \tag{2}$$

and the boundary conditions are prescribed by

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t), \qquad t \geqslant 0. \tag{3}$$

* Corresponding author.
E-mail address: rmohanty@maths.du.ac.in (R.K. Mohanty).

0096-3003/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2006.02.032

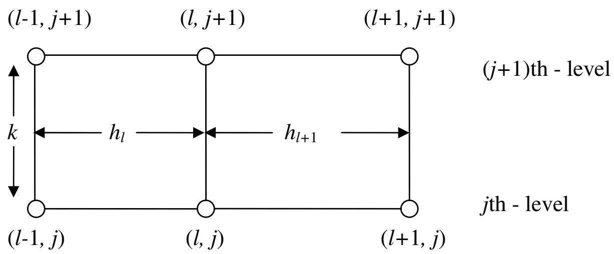

Fig. 1. Three-spatial grid points.

We assume that the function $\phi(x,t,u,u_x,u_t)$, $f(x)$, $g_0(t)$ and $g_1(t)$ are sufficiently smooth and their required high-order derivatives exist.

Many real field problems in science and engineering can be modeled by non-linear parabolic partial differential equations. Most non-linear parabolic partial differential equations defy closed form solutions because the usual elegant theory valid for their linear counterparts often fails for them. To facilitate provision of analytic solution, numerical techniques leading to the use of iterative methods are commonly employed.

For the numerical solution of (1), the non-linear Crank–Nicolson type scheme has enjoyed great popularity. It is well known that the accuracy of the Crank–Nicolson scheme on a uniform mesh is of order 2 in space and 2 in time. Using constant mesh, numerical solution of (1) has been considered by a few authors. Douglas and Jones [1], Reynolds [2] and Jacques [3] have discussed lower order convergent numerical methods for the solution of non-linear parabolic equation (1) on a uniform mesh. Difference methods of order 2 in time and 4 in space for the solution of general non-linear parabolic equation on a uniform grid have been suggested by Jain et al. [4,5], Mohanty [6] and Mohanty et al. [7]. Using variable mesh Jain et al. [8], Mohanty [9], Evans and Mohanty [10] and Mohanty and Khosla [11] have discussed various numerical methods for the solution of non-linear two point boundary value problems. It has been verified that Crank–Nicolson type variable mesh method for the solution of differential equation (1) is always unstable even for a simple heat conduction equation. In this paper, we derive a new stable two-level implicit method based on arithmetic average discretization on a variable mesh for the solution of non-linear parabolic equation (1). We use only three spatial grid points (see Fig. 1) and two evaluations of the function ϕ. The accuracy of the method is of $O(k^2h_l^{-1}+h_l^2)$, where $k>0$ and $h_l>0$, $l=1(1)N$ are grid spacing in time and space directions, respectively. Difficulties were experienced in the past for the numerical solution of parabolic equations in polar coordinates. The solution usually deteriorates in the vicinity of the singularity. The proposed variable mesh method is directly applicable to parabolic equations in polar coordinates. In this case, we do not require any special technique or special scheme to handle the singular problem near the boundary.

In Section 2, we give derivation of the proposed variable mesh method. Linear stability analysis for a linear difference scheme, which is consistent with diffusion equation is discussed in Section 3. In Section 4, we report results of numerical experiments to verify the computational efficiency of the proposed new variable mesh method. Final remarks are included in Section 5.

2. A new stable variable mesh method

For the numerical integration of the proposed initial boundary value problem (1)–(3), we discretize the solution region $\Omega\equiv\{(x,t)\,|\,0<x<1,t>0\}$. such that $0=x_0<x_1<\cdots<x_{N+1}=1$ and $t_{j+1}-t_j=k$, $j=0,1,\ldots$ Let $h_l=x_l-x_{l-1}>0$, $l=1(1)N+1$ and $k>0$ be the mesh sizes in x- and t-directions, respectively. Let $U_l^j=u(x_l,t_j)$ and u_l^j be the exact and approximate solution values of $u(x,t)$ at the grid point (x_l,t_j), respectively. The mesh ratio parameter is given by $\sigma_l=(h_{l+1}/h_l)>0$, $l=1(1)N$. When $\sigma_l=1$, it reduces to a constant mesh case. Our method is described as follows.

At the grid point (x_l,t_j), we define the following approximations:

Let

$$\bar{t}_j=t_j+\theta k, \tag{4}$$