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Some new iterative methods with three-order convergence

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Abstract

Some new iterative formulae having three-order convergence are discussed in this paper. The theoretical analysis and numerical experiments show that new iterative formulae are effective and comparable to well-known method of Newton and Stefensen method.

Keywords: Iterative formulae; Newton's method; Order of convergence

1. Introduction

We consider a family of iterative methods for computing approximate solutions of the nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f(x)$ is real value function.

Newton's method [2–4] is the well-known iterative algorithm to find solution of (1). However, Newton's method may fail to converge in case the initial guess is far from zero or the derivative is small in the vicinity of the required root.

Recently, Kanwar et al. [1], present a class of iterative formulae with quadratic convergence which can be used an alternative to Newton's method or in cases where Newton's method is not successful.

This paper proposes some new three-order convergence iteration formulae, which can also be used an alternative to Newton's method or in cases where Newton's method is not successful. Several numerical tests show that new formulae have advantages over Newton's method, Steffensen method [4,5].

2. Some three-order iteration formulae

Let x^* be the exact root of Eq. (1) and $x_0, x_1, x_2, \ldots, x_n$ be the known approximations for the required root.

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Consider the following auxiliary equations with parameter p:

$$g(x) = p^2(x - x_0)^2 f^2(x) - f^2(x) = 0, \tag{2}$$

where $p \in R$, $|p| < +\infty$.

It is clear that the root of Eq. (1) is also the roots of Eq. (2) and vice versa.

If $x_{n+1} = x_n + h$ be the better approximation for the required root, Eq. (2) gives

$$p^2 h^2 f^2(x_n + h) - f^2(x_n + h) = 0. \tag{3}$$

Expanding by Taylor's theorem and simplifying, we get

$$h = \frac{f(x_n)}{f'(x_n) + \operatorname{sign}(f'(x_n))f(x_n)}, \tag{4}$$

retaining the terms upto $o(h^2)$ excluding the term containing second derivative. Then the quadratical convergent iteration formulae presented by Kanwar et al. [1] are obtained:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \operatorname{sign}(f'(x_n))pf(x_n)}, \quad n = 0, 1, 2, \ldots \tag{5}$$

For the formula (5), we divide the following two cases to present.

Case 1. Consider the following the formulae:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + p(x)f(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $|p(x)| < +\infty$ is real function. Let $e_n = x_n - x^*$. Using the Taylor series expansion of $f(x)$, then we have

$$f(x_n) = f'(x^*)e_n + \frac{f''(x^*)}{2}e_n^2 + o(e_n^2),$$

$$f'(x_n) = f'(x^*) + f''(x^*)e_n + \frac{f'''(x^*)}{2}e_n^2 + o(e_n^2).$$

And

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{f''(x^*)}{2f'(x^*)}e_n^2 + \left(\frac{f''^2(x^*)}{2f'^2(x^*)} - \frac{f'''(x^*)}{3f'(x^*)}\right)e_n^3 + o(e_n^3).$$

So we obtain

$$\begin{aligned} e_{n+1} &= e_n\left(1 - \frac{1 - \frac{f''(x^*)}{2f'(x^*)}e_n + o(e_n)}{1 + p(x^*)\left(e_n - \frac{f''(x^*)}{2f'(x^*)}e_n^2 + \left(\frac{f''^2(x^*)}{2f'^2(x^*)} - \frac{f'''(x^*)}{3f'(x^*)}\right)e_n^3 + o(e_n^3)\right)}\right) \\ &= e_n^2\left(\frac{p(x^*) + \frac{f''(x^*)}{2f'(x^*)} + \left(\frac{f'''(x^*)}{3f'(x^*)} - \frac{f''^2(x^*)}{4f'^2(x^*)}\right)e_n + o(e_n)}{1 + p(x)\left(e_n - \frac{f''(x^*)}{2f'(x^*)}e_n^2 + \left(\frac{f''^2(x^*)}{2f'^2(x^*)} - \frac{f'''(x^*)}{3f'(x^*)}\right)e_n^3 + o(e_n^3)\right)}\right). \end{aligned}$$

So if taking $p(x) = -\frac{f''(x)}{2f'(x)}$, then formula (6) has at least three-order convergence, i.e.,

$$\lim_{n \to +\infty} \frac{e_{n+1}}{e_n^3} = \frac{f'''(x^*)}{3f'(x^*)} - \frac{f''^2(x^*)}{4f'^2(x^*)}, \tag{7}$$

we get one three-order formula

$$x_{n+1} = x_n - \frac{2f'(x_n)f(x_n)}{2f'^2(x_n) - f''(x)f(x_n)}, \quad n = 0, 1, 2, \ldots \tag{8}$$

And if noting $\lim_{x \to x^*} -\frac{f'(x+f(x)) - f'(x)}{2f(x)f'(x)} = -\frac{f''(x^*)}{2f'(x^*)}$, then, another three-order formula is obtained

$$x_{n+1} = x_n - \frac{2f'(x_n)f(x_n)}{2f'^2(x_n) + f'(x) - f'(x_n + f(x_n))}, \quad n = 0, 1, 2, \ldots \tag{9}$$