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On critical capacity of arcs in a directed network

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Abstract

A methodology is proposed to find the critical capacity of an arc of a feasible network wherein lower bounds on flows on the various arcs are non-negative integers.

Keywords: Directed network; Critical capacity; Flow problem

1. Introduction

Consider a directed network $G = (N, A)$ comprising of $n(=|N|)$ nodes and $m(=|A|)$ arcs. Arc $(i,j) \in A$ has a lower bound l_{ij} and an upper bound u_{ij} on the arc flow. When $l_{ij} = 0$ for all $(i,j) \in A$, the max flow from one node (say node 1) to the other node (say node n) is always finite. In such situations G is said to be a feasible network.

The critical capacity (denoted by k^*_{ij}) of an arc (i,j) is defined as the extent up to which flow can be increased on the arc inducing the raise in the max flow value. If $v(0)$ is the max flow value from the node 1 (source) to the node n (sink) when the flow on the arc (i,j) is set at $l_{ij} = 0$ and $v(\infty)$ is the value of the max flow when the flow on the arc (i,j) is allowed up to any large extent, then the critical capacity of the arc (i,j) is $k^*_{ij} = v(\infty) - v(0)$. That is, as the capacity of the arc (i,j) is raised from 0 to k^*_{ij} the value of the max flow from the source to the sink rises from $v(0)$ to $v(\infty)(= v(0 + k^*_{ij}))$. Further, raising the flow on (i,j) beyond k^*_{ij} will not raise the max flow value [1]. Thus, the max flow from the source to the sink

$$\begin{aligned} &= v(0 + \lambda), \quad 0 \leqslant \lambda \leqslant k^*_{ij}, \\ &= v(0 + k^*_{ij}) = v(\infty), \quad \lambda > k^*_{ij}. \end{aligned}$$

Further, whenever the capacity of the arc (i,j) is $k_{ij} < k^*_{ij}$, (i,j) is a forward arc in every minimum capacity cutset separating the source and the sink. If $k_{ij} > k^*_{ij}$, (i,j) is not a forward arc in any minimum capacity cutset separating the source and the sink. If $k_{ij} = k^*_{ij}$, there exists a minimum capacity cutset separating the source and the sink that contains (i,j) as a forward arc, and another that does not contain (i,j) as a forward arc [1].

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It is also known that 'destruction' of an arc (i,j) reduces the max flow value form the source to the sink by $\min(k_{ij}, k^*_{ij})$, where destroying an arc (i,j) means setting $u_{ij} = l_{ij}(=0)$ [1].

When $l_{ij} \geqslant 0$ and integer for all $(i,j) \in A$, then the max flow in G from the source to the sink may not exist. That is, G may not be a feasible network. The feasibility of G when $l_{ij} \geqslant 0\ \forall (i,j) \in A$ is examined by solving a circulation problem created by linking the sink node to the source node by the arc $(n,1)$ with lower bound $l_{n1} = 0$ and upper bound $u_{n1} = \infty$ on the arc flow. Existence of a feasible circulation is examined by finding the max flow from the new source s^* to the new sink t^* in the enlarged network $G^* = (N^*, A^*)$ where, $N^* = N \cup \{s^*, t^*\}$ and $A^* = A \cup \{(n,1)\} \cup \{(s^*, i), (i, t^*), i \in N\}$. The flow on the arc (i,j) in A^* has lower bound l^*_{ij} and upper bound u^*_{ij} defined as

$$
\begin{aligned}
& l^*_{ij} = 0 \quad \forall (i,j) \in A^*, \\
& u^*_{ij} = u_{ij} - l_{ij} \quad \forall (i,j) \in A, \\
& u^*_{s^*i} = \sum_{\substack{r \in N: \\ (r,i) \in A}} l_{ri}, \quad i \in N, \\
& u^*_{it^*} = \sum_{\substack{r \in N: \\ (i,r) \in A}} l_{ir}, \quad i \in N, \\
& u^*_{n1} = \infty.
\end{aligned}
$$

Suppose the max flow from s^* to t^* in G^* has the value V. If $V < \sum_{(i,j)\in A} l_{ij}$, then there does not exist a feasible circulation and G is termed as infeasible [1].

If $V = \sum_{(i,j)\in A} l_{ij}$, then there exists a feasible circulation and G is said to be a feasible network in the sense it will have a finite max flow from the source to the sink. In this case, arcs (s^*, i) and $(i, t^*), i \in N$ are saturated and there are only two minimum capacity cutsets separating the source and the sink: (i) $(X, \overline{X})$, where $X = \{s^*\}$, $\overline{X} = N^* \setminus \{s^*\}$ and (ii) $(X, \overline{X})$, where $X = N^* \setminus \{t^*\}$, $\overline{X} = \{t^*\}$ (Ref. Lemma 2.1).

No arc $(i,j) \in A$ will be a forward arc of any minimum capacity cutset separating the source and the sink with respect to the max flow vector of value $V = \sum_{(i,j)\in A} l_{ij}$.

In this paper, a feasible directed network G is studied in which the lower bound on the flow on its arcs is not necessarily zero. The study aims at finding the critical capacity of the arcs in such a network. It is found that it is not as simple as in the case when $l_{ij} = 0 \forall (i,j) \in A$ [2,1]. As per our knowledge, no literature on critical capacity when $l_{ij} \geqslant 0, (i,j) \in A$ is available.

In the next section on the theoretical development, 'modified lower bound' on the flow on an arc $(i,j) \in A$, the given lower bound being $l_{ij} > 0$, is defined. A theorem is established which helps in the computation of such lower bounds. Latter part of this section deals with the critical capacity of an arc $(i,j) \in A$. In the last section, two numerical illustrations are given in support of the developed theory.

2. Theoretical development

Let $G = (N, A)$ be a directed network and $l_{ij} \geqslant 0$, u_{ij}, respectively be the lower bound and the upper bound on the flow on an arc $(i,j) \in A$.

Let $v(l_{ij} + \lambda)$ be the max flow in G from the source to the sink when the upper bound on the flow on the arc (i,j) is set equal to $(l_{ij} + \lambda)$.

Definition 2.1 (*Modified Lower Bound*). Suppose $v(l_{ij} + \lambda)$ does not exist for $0 \leqslant \lambda < L_{ij}$. If $v(l_{ij} + L_{ij})$ exists, then $\hat{l}_{ij} = l_{ij} + L_{ij}$ is called the modified lower bound on the flow on the arc (i,j).

Lemma 2.1. *Whenever the max flow in G^* is $\sum_{(i,j)\in A} l_{ij}$, there exist ONLY two minimum capacity cutsets separating the source and the sink namely*:

(i) $(X, \overline{X})$, *where* $X = \{s^*\}$, $\overline{X} = N^* \setminus \{s^*\}$ *and*
(ii) $(X, \overline{X})$, *where* $X = N^* \setminus \{t^*\}$, $\overline{X} = \{t^*\}$.