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A class of non-uniform mesh three point arithmetic average discretization for $y'' = f(x, y, y')$ and the estimates of y'

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Abstract

We propose two new non-uniform mesh three point arithmetic average discretization strategy of order two and three, to solve non-linear ordinary differential equation $y'' = f(x, y, y')$, $a < x < b$ and the estimates of first-order derivative y', where $y = y(x)$, subject to the essential boundary conditions $y(a) = A$, $y(b) = B$. Both methods are compact and directly applicable to singular problems. There is no need to discuss any special scheme for the singular problems. Error analysis of a proposed method is discussed. Numerical experiments are performed to study the convergence behaviors and efficiency of the proposed methods.

Keywords: Non-uniform mesh; Non-linear equation; Singular problem; Diffusion–convection equation; Burgers' equation, Estimation; RMS errors

1. Introduction

In this article, we study some effective numerical techniques to solve the non-linear ordinary differential equation of the form

$$y'' = f(x, y, y'), \quad a < x < b \tag{1.1}$$

over a non-uniform mesh subject to essential boundary conditions

$$y(a) = A, \quad y(b) = B, \tag{1.2}$$

where A, B are finite constants. The details of the existence and uniqueness conditions of the above boundary value problem (1.1) and (1.2) are discussed in [1].

In 1984, Bogucz and Walker [2] gave a five-point uniform mesh discretization using two function evaluations for each interior grid point. However, in their method even though the number of function evaluation was two, the Jacobian of the resulting non-linear system for the numerical solution is pentadiagonal, and thus

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each application of Newton method requires the solution of a five-diagonal linear system. Later, Fox and Mayers [3] described a three-point uniform mesh discretization which involves three function evaluations for each internal mesh points and discussed its applications to differential-eigenvalue problems. Jain et al. [4] have discussed a class of three-point non-uniform mesh numerical methods which involves four function evaluations. However, their methods are not applicable to singular problems. Later, Mohanty et al. [5], Evans and Mohanty [6] and Mohanty [7] have developed a family of variable mesh methods and the estimates of first-order derivative for the solution of boundary value problem (1.1) and (1.2). Their methods involve only three non-uniform grid points and a special technique is required to solve singular problems. For linear differential equations, all these methods form tri-diagonal linear systems, whereas for non-linear cases, the Jacobian of these systems are tri-diagonal and therefore each application of Newton method requires the solution of a tri-diagonal system. Chawla and Shivkumar [8] have described a new fourth-order accurate three point arithmetic average discretization on a uniform mesh for the differential equation (1.1) which is more efficient and directly applicable to singular problems without any modification. To the author's knowledge no third-order accurate three point arithmetic average variable mesh discretization for the differential equation (1.1) is known in the literature so far.

In the present paper, we propose more efficient non-uniform mesh three point discretization of order two and three for the general non-linear differential equation (1.1), using just two and three function evaluations, respectively. Together with the boundary values, our present methods lead to non-linear systems whose Jacobian are tri-diagonal. Thus, if the differential equation (1.1) is linear, our methods lead to the solution of a tri-diagonal linear system; while for non-linear problems, each application of Newton method requires the solution of a tri-diagonal linear system. In addition, we also discuss second and third-order non-uniform mesh methods for the estimates of $(\mathrm{d}y/\mathrm{d}x)$, in more general case, which are of quite often interest in many applied problems. In next section, we give complete derivation of proposed numerical methods. In Section 3, we discuss convergence theory and in Section 4, we provide numerical results of some model problems to illustrate the proposed numerical methods and their convergence.

2. Derivation of numerical methods

We discretize the solution region $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$. Let $h_k = x_k - x_{k-1}$, $h_{k+1} = x_{k+1} - x_k$ and the mesh ratio $\sigma_k = h_{k+1}/h_k > 0$. When $\sigma_k = 1$, it reduces to the constant mesh case. Let $Y_k = y(x_k)$ and $Z_k = y'(x_k)$ be the exact solution values of y and y' at the mesh point x_k and y_k and z_k be their approximate solution values, respectively.

We start with the following three point arithmetic average discretization for $y'' = f$:

$$Y_{k+1} - (1+\sigma_k)Y_k + \sigma_k Y_{k-1} = \frac{\sigma_k h_k^2}{2}\left[\frac{(1+2\sigma_k)}{3} f_{k+1/2} + \frac{(2+\sigma_k)}{3} f_{k-1/2}\right] + T_k^{(2)} \tag{2.1}$$

and

$$Y_{k+1} - (1+\sigma_k)Y_k + \sigma_k Y_{k-1} = \frac{\sigma_k h_k^2}{3}\left[\sigma_k f_{k+1/2} + \frac{(1+\sigma_k)}{2} f_k + f_{k-1/2}\right] + T_k^{(3)}, \tag{2.2}$$

where

$$T_k^{(2)} = \mathrm{O}(h_k^4) \text{ and } T_k^{(3)} = \mathrm{O}(h_k^5) \quad \text{for } \sigma_k \neq 1.$$

Now, let

$$\overline{Y}_{k+1/2} = \frac{1}{2}(Y_{k+1} + Y_k), \tag{2.3a}$$

$$\overline{Y}_{k-1/2} = \frac{1}{2}(Y_{k-1} + Y_k), \tag{2.3b}$$

$$\overline{Y}'_{k+1/2} = \frac{1}{\sigma_k h_k}(Y_{k+1} - Y_k), \tag{2.3c}$$