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A modification of pseudo-spectral method for solving a linear ODEs with singularity

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Abstract

In this paper, first we introduce, briefly, pseudo-spectral method for numerical solution of ODE's and focus on those problems in which some of coefficient functions or solution function are singular. Then, by expressing weak and strong aspects of spectral methods to solve these kind of problems, a modified pseudo-spectral method which is more efficient than other spectral methods is suggested. We compare the methods with some numerical examples.

Keywords: Spectral method; ODE; Singularity; Pseudo-spectral method

1. Introduction

The spectral methods arise from the fundamental problem of approximation of a function by interpolation on an interval, and are very much successful for the numerical solution of ordinary or partial differential equations [1]. Since the time of Fourier (1882), spectral representations in analytic study of differential equations are used and their applications for numerical solution of ordinary differential equations refers, at least, to the time of Lanczos [2]. A survey of some applications is given in [3].

Spectral methods may be viewed as an extreme development of the class of discretization scheme for differential equations known generally as the *method of weighted residuals* (MWR) [5]. The key elements of the MWR are the trial functions (also called approximating functions) which are used as basis functions for a truncated series expansion of the solution, and the test functions (also known as weight functions) which are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. The choice of such functions distinguishes between the three most commonly used spectral schemes, namely, Galerkin, collocation (also called pseudo-spectral) and Tau version. The Tau approach is a modification of Galerkin method that is applicable to problems with non-periodic boundary conditions. In broad

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terms, Galerkin and Tau methods are implemented in terms of the expansion coefficients [4], whereas collocation methods are implemented in terms of physical space values of unknown function.

The basic idea of spectral methods to solve differential equations is to expand the solution function as a finite series of very smooth basis functions, as given

$$y^N(x) = \sum_{k=0}^{N} a_k \phi_k(x), \tag{1.1}$$

in which, the best choice of ϕ_k, are the eigenfunctions of a singular Sturm–Liouville problem. If the function y belongs to $C^\infty[a,b]$, the produced error of approximation (1.1), when N tends to infinity, approaches zero with exponential rate [1]. This phenomenon is usually referred to as "spectral accuracy". [3]. The accuracy of derivatives obtained by direct, term by term, differentiation of such truncated expansion naturally deteriorates [1], but for low order derivatives and sufficiently high- order truncations this deterioration is negligible. So, if solution function and coefficient functions are analytic on $[a,b]$, spectral methods will be very efficient and suitable.

2. Pseudo-spectral method

In this section, we briefly introduce pseudo-spectral method. For this reason, first we consider the following differential equation:

$$Ly = \sum_{i=0}^{m} f_{m-i}(x) D^i y = f(x), \quad x \in [-1,1], \tag{2.1}$$

$$Ty = C, \tag{2.2}$$

where f_i, $i = 0, 1, \ldots, m, f$, are known real functions of x, D^i denotes ith order of differentiation with respect to x, T is a linear functional of rank N and $C \in \mathbf{R}^m$.

Here (2.2) can be initial, boundary or mixed conditions. The basic of spectral methods to solve this class of equations is to expand the solution function, y, in (2.1) and (2.2), as a finite series of very smooth basis functions, as given below:

$$y^N(x) = \sum_{k=0}^{N} a_k T_k(x), \tag{2.3}$$

where, $\{T_k(x)\}_0^k$ is the sequence of Chebyshev polynomials of the first kind. With replacing y by y^N in (2.2), we define residual term by $r^N(x)$ as follows:

$$r^N(x) = Ly^N - f. \tag{2.4}$$

In spectral methods, the main target is to minimize $r^N(x)$ as much as possible with regard to (2.2). Implementation of these methods lead to a system of linear equations with $N+1$ equations and $N+1$ unknowns $a_0, a_1, \ldots, a_N$.

In the rest of this section, we discuss only one of the three spectral methods, namely, collocation (also known as pseudo-spectral) method, also we use Tau method for numerical solution of second order linear differential equations to compare the results with pseudo-spectral method. It is to be noted that this discussion can be extended to the general form Eqs. (2.1) and (2.2).

Consider the following differential equation:

$$\begin{aligned} &P(x)y'' + Q(x)y' + R(x)y = S(x), \quad x \in (-1,1), \\ &y(-1) = a, \quad y(1) = b. \end{aligned} \tag{2.5}$$

First, for an arbitrary natural number N, we suppose that the approximate solution of Eq. (2.5) is given by (2.3), where $\underline{a} = (a_0, a_1, \ldots, a_N)^t \in \mathbf{R}^{N+1}$ is the coefficients vector and $\{T_k(x)\}_0^k$ is the sequence of Chebyshev polynomials of the first kind. Now if we put

$$V(x) = \sum_{k=0}^{N} a_k T_k(x), \tag{2.6}$$